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CONTROLLABILITY AND STABILIZATION OF PROGRAMMED MOTIONS OF AN AUTOMOBILE-TYPE TRANSPORT ROBOT†

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A non-linear model of the motion of an automobile-type transport robot (TR) with absolutely rigid wheels, a steering device and actuators based on DC motors, is considered. Such a model for TR motion is a non-holonomic electromechanical system and, if the dynamics of the actuators and the steering device (forces of elasticity and attenuation in its elements) is ignored, corresponds to the model of automobile motion devised by Lineikin [1]. Non-linear canonical transformations of the state and control space coordinates are constructed which reduce the initial equations of motion of the TR to a simpler canonical form, convenient for the analysis and synthesis of control systems for the TR. These transformations are used to find the conditions for the controllability of the TR as a controlled object. Algorithms are given for constructing programmed controls and programmed motions of the TR. Stabilizing control laws are synthesized that make the programmed motions of the TR asymptotically stable and guarantee that the transients will have preassigned properties. © 2003 Elsevier Ltd. All rights reserved.

1. EQUATIONS OF THE MATHEMATICAL MODEL OF THE MOTION OF A TRANSPORT ROBOT. STATEMENT OF THE PROBLEM

1. We consider a model for the motion of an automobile-type transport robot (TR), which, as an electromechanical system, consists of several interlinked components: a four-wheeled chassis with a body, front and rear bridges, absolutely rigid wheels, a steering device whose elements admit of elastic deformation, and electrical actuators based on independently activated DC motors whose mechanisms for transmitting the motion (the transmissions) have absolutely rigid elements.

The body of the TR consists of the body of the rear bridge and a longitudinal beam rigidly linking the body of the rear bridge with the fixed part of a mechanism for turning the front bridge.

We shall assume that the TR is dynamically symmetric, that is, the centre of mass C of the body of the TR lies on the longitudinal axis BA passing through the midpoints A and B of the axles A_1AA_2 and B_1BB_2 of the front and rear wheels, respectively (see Fig. 1).

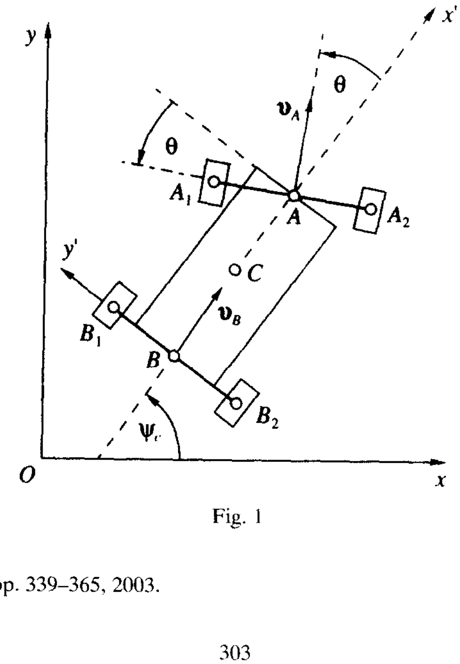

Fig. 1

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It will be assumed that the motion of each wheel can be considered as pure rolling without slipping, both longitudinally, in the plane of the wheels, and transversely, perpendicular to the plane of the wheels. Under these assumptions $\mathbf{v}_A$ and $\mathbf{v}_B$, the velocity vectors of the midpoints A and B of the front and back axles, are parallel at each instant of time to the planes of the corresponding wheels; that is, $\mathbf{v}_B$, the velocity vector of the point B, will always point along the longitudinal axis BA and the velocity vector $\mathbf{v}_A$ of the point A will always be at an angle θ to that axis.

When investigating the special features of plane-parallel motion of this TR model, we shall consider a simplified scheme of the model of TR motion of automobile type, devised by Lineikin [1] (see also [2, pp. 22–32] and later refined by Lobas [3, pp. 98–109].

Under the aforementioned simplifying assumptions, previously established relations ([3], p. 105, Eqs (5.24); p. 107, Eqs (5.34); p. 109, Eqs (5.41)) enable us to present the equations of motion of this TR model, relative to some fixed Cartesian system of coordinates (CSC) $\Sigma = Oxyz$, in the form

$$\dot{x}_c = \dot{\psi}_c(l\,\mathrm{ctg}\,\theta\cos\psi_c - l_2\sin\psi_c)$$

$$\dot{y}_c = \dot{\psi}_c(l\,\mathrm{ctg}\,\theta\sin\psi_c + l_2\cos\psi_c)$$

$$A_0(\theta)\begin{Vmatrix}\ddot{\psi}_c\\ \ddot{\theta}\end{Vmatrix} + b_0(\theta, \dot{\psi}_c, \dot{\theta}) = D_0(\theta)Q_u \tag{1.1}$$

$$J_{ri}\ddot{\alpha}_i + k_{f1i}\dot{\alpha}_i + i_{pi}^{-1}\eta_{pi}^{-1}Q_{ui} = k_{mi}I_{ai}$$

$$L_{ai}\dot{I}_{ai} + R_{ai}I_{ai} + k_{ei}\dot{\alpha}_i = u_{ai}, \quad i = 1, 2$$

where

$$A_0(\theta) = \begin{Vmatrix} J_0 + ml^2\mathrm{ctg}^2\theta & \Theta_2 \\ \Theta_2 & \Theta_2 \end{Vmatrix} = \|a_{0ij}(\theta)\|_{i,j=1,2}$$

$$b_0(\theta, \dot{\psi}_s, \dot{\theta}) = \begin{Vmatrix} -\dfrac{ml^2\mathrm{ctg}\,\theta}{\sin^2\theta}\dot{\psi}_\varepsilon\dot{\theta} + F_c(\theta, \dot{\psi}_c)l\,\mathrm{ctg}\,\theta \\ -k_{y2}\theta - k_{f21}\dot{\theta} - k_{f22}\dot{\theta}^3 \end{Vmatrix} \tag{1.2}$$

$$D_0(\theta) = \mathrm{diag}\left(\frac{l\,\mathrm{ctg}\,\theta}{r}, 1\right), \quad J_0 = J + ml_2^2 + 2l_1l_2m_2$$

x_c, y_c are the coordinates, and $\dot{x}_c, \dot{y}_c$ are the velocities of the centre of mass M of the TR in the fixed CSC Oxy, ψ_c is the course angle – the angle of inclination (turn) of the longitudinal axis BA of the RT to the Ox axis, θ is the angle of rotation of the front wheels, measures from the direction of the longitudinal axis BA of the TR, it is assumed that a leftward rotation of the wheels corresponds to positive values of the angles ψ_c and θ, a dot over a symbol denotes the operation of differentiation with respect to time t; $l = l_1 + l_2$ is the length of the base of the body, the segments $BA = l_1$ and $BC = l_2$ are the distances from the centre of mass C of the TR to its front and back axles, $m = m_1 + m_2$ is the mass of the TR, where m_1 is the mass of the body including the masses of the wheels, $\Theta_1 = J_1 + m_1l_2^2$ and J_1 are the moments of inertia of the body together with the wheels about a vertical axis through the points B and C, respectively, that is, J_1 is the central moment of inertia of the body, m_2 and Θ_2 are the mass and moment of inertia of the front axle with the steering device together with the front wheels about a vertical axis through the point A, $\Theta = \Theta_1 + \Theta_2 + m_2l^2$ is the moment of inertia of the TR about a vertical axis through the point B, $J = J_1 + \Theta_2 + m_2l_1^2$ is the moment of inertia of the TR about a vertical axis Cz through the point C, A_0 is a symmetric positive-definite matrix of order 2×2, b_0 is a two-dimensional vector function, D_0 is a diagonal 2×2 matrix-valued function

$$\begin{Vmatrix}\dot{\alpha}_1\\ \dot{\alpha}_2\end{Vmatrix} = \kappa_0(\theta)\begin{Vmatrix}\dot{\psi}_c\\ \dot{\theta}\end{Vmatrix}, \quad \kappa_0(\theta) = \mathrm{diag}(l\,\mathrm{ctg}\,\theta\, i_{p1}r^{-1}, i_{p2}) \tag{1.3}$$

α_i is the angle of rotation of the shaft of the ith DC motor

$$V_B = V_{cx'} = (\mathbf{v}_c)_{x'} = r(\dot{q}_1 + \dot{q}_2)/2 = i_{p1}^{-1}\dot{\alpha}_1 r = \\ = l\operatorname{ctg}\theta\dot{\psi}_c = \rho_B\dot{\psi}_c = \kappa_B^{-1}\dot{\psi}_c \quad (\rho_B = \kappa_B^{-1} = l\operatorname{ctg}\theta) \tag{1.4}$$

is the velocity of the point $B(x_B, y_B, r)_\Sigma$, equal to the projection V_{cx} of the velocity vector v_c of the centre of mass of $C(l_2, 0)_{\Sigma'}$ of the TR onto the axis Bx' (directed along the longitudinal axis BA of the TR toward the front part of the body of the TR) of the moving CCS $\Sigma' = Bx'y'$, it is assumed that if $V_B > 0$ forward motion of the TR takes place in a direction that coincides with that of the Bx' axis, but if $V_B < 0$, the direction is opposed to that of the Bx' axis, r and $\dot{q}_1$ and $\dot{q}_2$ are the radius and angular velocities of the wheels of the rear bridge of the TR chassis, respectively, $\rho_B = l\operatorname{ctg}\theta$ is the radius of curvature $\kappa_B = \rho_B^{-1} = (l\operatorname{ctg}\theta)^{-1}$ of the trajectory of motion of the TR at the point B, i_{pi} and η_{pi} are the coefficients of transmission and efficiency of the ith transmission reductor

$$F_c(\theta, \dot{\psi}_c) = k_{fV1} l\operatorname{ctg}\theta\dot{\psi}_c + k_{fV2}(l\operatorname{ctg}\theta\dot{\psi}_c)^2 + F_{c0} = \\ = \bar{F}_c(V_B) = k_{fV1}V_B + k_{fV2}V_B^2 + F_{c0}$$

is the force of resistance to forward motion of the TR, $k_{fV1} \geqslant 0$ and $k_{fV2} \geqslant 0$ are the attenuation coefficients, $F_{c0} \geqslant 0$ is a constant, $Q_\theta = -k_{y2}\theta - k_{f21}\dot{\theta} - k_{f22}\dot{\theta}^3$ is a generalized force allowing for the forces of elasticity and attenuation acting on the elements of the steering device [3, p. 109], k_{y2} and k_{f21}, k_{f22} are the stiffness of the steering device and attenuation coefficients, Q_{u1} and Q_{u2} are the components of the two-dimensional vector

$$Q_u = \operatorname{col}(Q_{u1}, Q_{u2}) \tag{1.5}$$

of the generalized (rotating) torques Q_{u1} and Q_{u2} conveyed from the motor shafts through the transmission to the wheels of the front axis and to the steering device, respectively,

$$P_u = Q_{u1}/r \tag{1.6}$$

is the force acting along the longitudinal axis BA of the TR in the direction of Bx' axis,

$$I_a = \operatorname{col}(I_{a1}, I_{a2}) \tag{1.7}$$

is the two-dimensional vector of the currents I_{a1} and I_{a2} in the armature circuits of the DC motors, J_{ri} is the moment of inertia of the rotor of the ith motor, k_{f1i} is the coefficient of the moment of resistance of viscous friction $M_{ci} = -k_{f1i}\dot{\alpha}_i$ on the shaft of the ith motor, k_{mi} is the coefficient of the electromagnetic torque $M_i = k_{mi}I_{ai}$ of the ith motor, L_{ai} and R_{ai} are the total inductance and resistance of the armature circuit of the ith motor, k_{ei} is coefficient of proportionality of the back emf $u_{ei} = k_{ei}\dot{\alpha}_i$ of the ith motor

$$u_a = \operatorname{col}(u_{a1}, u_{a2}) \tag{1.8}$$

is the two-dimensional vector of the voltages u_{a1} and u_{a2} supplied to the armature circuits of the DC motor.

Note that, since the first two equations in the system of equations of motion (1.1) of the TR describe non-holonomic constraints [3, p. 105, Eqs (5.24)] between the wheel chassis and the supporting horizontal surface (realized by the wheels of the chassis), it follows that model (1.1) for the motion of the TR is a non-holonomic electromechanical system.

Eliminating the variables Q_{u1}, Q_{u2}, α_1, α_2 from Eqs (1.1), and also using relations (1.2)–(1.8), we obtain the equations of motion of the TR in the form of the following system of non-linear ordinary differential equations (ODEs)

$$\dot{\bar{z}} = \bar{F}(\bar{z}, u_a), \quad \bar{z}_0 = \bar{z}(t_0), \quad t \geq t_0 \tag{1.9}$$

where

$$\bar{z} = \operatorname{col}(\bar{z}_1, \bar{z}_2, \bar{z}_3, \bar{z}_4) \\ (\bar{z}_1 = \operatorname{col}(x_c, y_c), \bar{z}_2 = \operatorname{col}(\psi_c, \theta), \bar{z}_3 = \dot{\bar{z}}_2 = \operatorname{col}(\dot{\psi}_c, \dot{\theta}), \bar{z}_4 = I_a) \tag{1.10}$$

is the state vector of the TR, $\bar{z}_i = col(\bar{z}_{i1}, \bar{z}_{i2})$ $(i = 1, \dots, 4)$, and

$$\bar{F}(\bar{z}, u_a) = \text{col}(\bar{F}_1(\bar{z}_2^{31}), \bar{F}_2(\bar{z}_3), \bar{F}_3(\bar{z}_{22}^4), \bar{F}_4(\bar{z}_{22}^4, u_a)) \tag{1.11}$$

$$\begin{aligned}
\bar{F}_1(\bar{z}_2^{31}) &= \text{col}(\dot{\psi}_c(l\,\text{ctg}\,\theta\cos\psi_c - l_2\sin\psi_c), \dot{\psi}_c(l\,\text{ctg}\,\theta\sin\psi_c + l_2\cos\psi_c)) \\
\bar{F}_2(\bar{z}_3) &= \bar{z}_3 = \text{col}(\dot{\psi}_c, \dot{\theta}) \\
\bar{F}_3(\bar{z}_{22}^4) &= \bar{C}_3(\theta, \dot{\psi}_c, \dot{\theta}) + \bar{D}_3(\theta)I_\alpha \\
\bar{F}_4(\bar{z}_{22}^4, u_a) &= \bar{C}_4(\theta, \dot{\psi}_c, \dot{\theta}, I_a) + \bar{D}_4 u_a
\end{aligned} \tag{1.12}$$

are vector functions, where

$$\begin{aligned}
&\bar{z}_2^{31} = \text{col}(\bar{z}_2, \bar{z}_{31}), \quad \bar{z}_{22}^4 = \text{col}(\bar{z}_{22}, \bar{z}_3, \bar{z}_4) \\
&\bar{C}_3(\theta, \dot{\psi}_c, \dot{\theta}) = -A^{-1}(\theta)b(\theta, \dot{\psi}_c, \dot{\theta}), \quad \bar{D}_3(\theta) = A^{-1}(\theta)k_m \\
&A(\theta) = J_r\kappa_0(\theta) + i_p^{-1}\eta_p^{-1}D_0^{-1}(\theta)A_0(\theta) = \|a_{ij}(\theta)\|_{i, j = 1, 2} \\
&b(\theta, \dot{\psi}_c, \dot{\theta}) = (J_r\dot{\kappa}_0(\theta) + k_{f1}\kappa_0(\theta))\begin{Vmatrix}\dot{\psi}_c \\ \dot{\theta}\end{Vmatrix} + i_p^{-1}\eta_p^{-1}D_0^{-1}(\theta)b_0(\theta, \dot{\psi}_c, \dot{\theta}) \\
&\bar{C}_4(\theta, \dot{\psi}_c, \dot{\theta}, I_a) = L_a^{-1}\left[-R_aI_a - k_e\kappa_0(\theta)\begin{Vmatrix}\dot{\psi}_c \\ \dot{\theta}\end{Vmatrix}\right], \quad \bar{D}_4 = L_a^{-1}
\end{aligned} \tag{1.13}$$

$J_r, i_p, \eta_p, k_{f1}, k_m, L_a, R_a, k_e$ are diagonal 2×2 matrices with diagonal elements $J_{ri}, i_{pi}, \eta_{pi}, k_{f1i}, k_{mi}, L_{ai}, R_{ai}, k_{ei}$ $(i = 1, 2)$, respectively.

Applying non-linear one-to-one continuously differentiable transformations of the coordinates of the state space $\bar{z}$ (1.10) and $\hat{z}$ (6.18), that is

$$\begin{aligned}
&\hat{z} = \text{col}(\hat{z}_1, \hat{z}_2, \hat{z}_3, \hat{z}_4) \\
&(\hat{z}_1 = \text{col}(x_B, y_B), \hat{z}_2 = \text{col}(\psi_c, \kappa_B), \hat{z}_3 = \text{col}(V_B, \dot{\kappa}_B), \hat{z}_4 = \hat{I}_a)
\end{aligned} \tag{1.14}$$

and control u_a (1.8) and $\hat{u}_a$ (6.19), that is,

$$\hat{u}_a = \text{col}(\hat{u}_{a1}, \hat{u}_{a2}) \tag{1.15}$$

as defined by the formulae (see Appendix, Section 6)

$$\hat{z} = \Psi_0(\bar{z}), \quad \bar{z} \in \Omega_{\Psi 0} \tag{1.16}$$

$$\bar{z} = \Psi_0^{-1}(\hat{z}) = \Phi_0(\hat{z}), \quad \hat{z} \in \Omega_{\Phi 0} \tag{1.17}$$

and

$$\hat{u}_a = \Psi_{05}(\bar{z}_2^4, u_a) \quad (\bar{z}_2^4, u_a) \in \Omega_{\Psi 05} \tag{1.18}$$

$$u_a = \Psi_{05}^{-1}(\bar{z}_2^4, \hat{u}_a) = \Psi_{05}^{-1}(\Phi_{02}^4(\hat{z}_2^4), \hat{u}_a) = \Phi_{05}(\hat{z}_2^4, \hat{u}_a) \quad (\hat{z}_2^4, \hat{u}_a) \in \Omega_{\Phi 05} \tag{1.19}$$

respectively (where $\bar{z}_2^4$, = col $(\bar{z}_2, \bar{z}_3, \bar{z}_4,)$, $\hat{z}_2^4$, = col$(\hat{z}_2, \hat{z}_3, \hat{z}_4)$, $\Phi_{02}^4(\hat{z}_2^4)$ = col$(\Phi_{02}(\hat{z}_2), \Phi_{03}(\hat{z}_2^3), \Phi_{04}(\hat{z}_2^4)$, the vector functions $\Psi_0(\bar{z})$ and $\Phi_0(\hat{z})$ of the form (6.37) and (6.38), respectively, are defined on the respective sets $\Omega_{\psi 0}$ (6.39) and $\Omega_{\Phi 0}$ (6.40), and the vector functions $\Psi_{05}(\bar{z}_2^4, u_a)$ and $\Phi_{05}(\hat{z}_2^4, \hat{u}_a)$ of the form (6.41) and (6.42), respectively, are defined on the respective sets $\Omega_{\Psi 05}$ (6.43) and $\Omega_{\Psi 05}$ (6.44)), we reduce the equations of motion (1.9)–(1.13) and (1.8) of the TR to a simpler system of non-linear ODEs of the special form

$$\dot{\hat{z}} = \hat{F}(\hat{z}, \hat{u}_a), \quad \hat{z}_0 = \hat{z}(t_0), \quad t \geq t_0 \tag{1.20}$$

$\hat{z}$ being the state vector (1.14) of the system

$$\hat{F}(\hat{z}, \hat{u}_a) = J_{\Psi_0}((\Phi_0(\hat{z})) \cdot \bar{F}(\Phi_0(\hat{z}), \Phi_{05}(\hat{z}_2^4, \hat{u}_a))) = \\ = \mathrm{col}(\hat{F}_1(\hat{z}_{21}, \hat{z}_{31}), \hat{F}_2(\hat{z}_{22}^3), \hat{F}_3(\hat{z}_4), \hat{F}_4(\hat{u}_a)) \tag{1.21}$$

where $J_{\Psi_0}(\bar{z}) = \partial\Phi_0(\bar{z})/(\partial\bar{z})$ is the 8×8 Jacobian

$$\hat{F}_1(\hat{z}_{21}, \hat{z}_{31}) = \mathrm{col}(\hat{z}_{31}\cos\hat{z}_{21}, \hat{z}_{31}\sin\hat{z}_{21}) = \mathrm{col}(V_B\cos\psi_c, V_B\sin\psi_c) \\ \hat{F}_2(\hat{z}_{22}^3) = \mathrm{col}(\hat{z}_{22}\hat{z}_{31}, \hat{z}_{32}) = \mathrm{col}(\kappa_B V_B, \dot{\kappa}_B) \\ \hat{F}_3(\hat{z}_4) = \hat{z}_4 = \hat{I}_a, \quad \hat{F}_4(\hat{u}_a) = \hat{u}_a \tag{1.22}$$

are two-dimensional vector functions, $\hat{z}_{22}^3 = \mathrm{col}(\hat{z}_{22}, \hat{z}_3)$, and the first two equations describe non-holonomic constraints [4].

Incidentally, a particular version of such a system of equations for a model of TR motion – an automobile (in which the dynamics of the steering device and actuators, as well as the forces of elasticity and attenuation acting on the elements of the steering device, are ignored) – was described previously in [5, p. 20, Eqs (1)–(5)].

We shall assume that the auxiliary control impulses $\hat{u}_{a1}$ and $\hat{u}_{a2}$ (1.15) are such that

$$\ddot{\hat{u}}_{a1} = u_1, \quad \hat{u}_{a2} \equiv u_2 \tag{1.23}$$

where u_1 and u_2 are the components of the vector of controls

$$u = \mathrm{col}(u_1, u_2) \tag{1.24}$$

supplied to the inputs of system (1.20)–(1.23), (1.14), (1.15).

Then the equations of the model of TR motion of the special form (1.20)–(1.24), (1.14), (1.15) [referred to henceforth as the sp-model], may be written as a system of non-linear ODEs

$$\dot{z} = F(z, u), \quad z_0 = z(t_0), \quad t \geq t_0 \tag{1.25}$$

where

$$z = \mathrm{col}(z_1, \ldots, z_5) \;\; (z_1 = \hat{z}_1 = \mathrm{col}(x_B, y_B), z_2 = \mathrm{col}(V_B, \psi_c) \\ (V_B = \hat{z}_{31}, \psi_c = \hat{z}_{21}), \quad z_3 = \mathrm{col}(\hat{I}_{a1}, \kappa_B) \quad (\hat{I}_{a1} = \hat{z}_{41}, \kappa_B = \hat{z}_{22}) \\ z_4 = \mathrm{col}(\hat{u}_{a1}, \bar{\kappa}) \;\; (\bar{\kappa} = \dot{\kappa}_B = \hat{z}_{32}), \quad z_5 = \mathrm{col}(\tilde{u}_{a1}, \hat{I}_{a2}) \quad (\tilde{u}_{a1} = \dot{\hat{u}}_{a1}, \hat{I}_{a2} = \hat{z}_{42})) \tag{1.26}$$

is the state vector of the TR, $z_i = \mathrm{col}(z_{i1}, z_{i2}), z_i^j = \mathrm{col}(z_i, z_{i+1}, \ldots, z_j), j \geqslant i; z_i^i = z_i$, and

$$F(z, u) = \mathrm{col}(F_1(z_2), F_2(z_{21}, z_3), F_3(z_4), F_4(z_5), F_5(u)) \tag{1.27}$$

$$F_1(z_2) = \mathrm{col}(z_{21}\cos z_{22}, z_{21}\sin z_{22}) = \mathrm{col}(V_B\cos\psi_c, V_B\sin\psi_c) \\ F_2(z_{21}, z_3) = D_2(z_{21})z_3 = \mathrm{col}(\hat{I}_{a1}, \kappa_B V_B), \quad D_2(z_{21}) = \mathrm{diag}(1, z_{21}) \\ F_3(z_4) = z_4 = \mathrm{col}(\hat{u}_{a1}, \bar{\kappa}), \quad \bar{F}_4(z_5) = z_5 = \mathrm{col}(\tilde{u}_{a1}, \hat{I}_{a2}), \quad F_5(u) = u \tag{1.28}$$

are vector functions.

Note that the state vector z (1.26) of system (1.25)–(1.28), (1.24) is related to the state vector $\bar{z}$ of the original equations of TR motion (1.9)–(1.13) and (1.8) by non-linear transformations of the form

$$z = H_1\hat{z} + H_0\tilde{\hat{u}}_a = H_1\Psi_0(\bar{z}) + H_0\tilde{\hat{u}}_a \quad (\hat{z} = \Psi_0(\bar{z})) \tag{1.29}$$

$$\bar{z} = \Phi_0(\hat{z}) = \Phi_0(H_2 z) \quad (\hat{z} = H_2 z) \tag{1.30}$$

where

$$\tilde{u}_a = \mathrm{col}(\tilde{u}_{a1}, \tilde{u}_{a2}) = \mathrm{col}(\hat{u}_{a1}, \tilde{u}_{a1}) = \mathrm{col}(z_{41}, z_{51})$$
$$(\tilde{u}_{a1} = \hat{u}_{a1} = z_{41}, \tilde{u}_{a2} = \tilde{u}_{a1} = \dot{\hat{u}}_{a1} = z_{51}) \tag{1.31}$$

and H_1, H_0 and H_2 are constant matrices of the respective orders 10×8, 10×2 and 8×10, whose elements are respectively

$$h_{111} = h_{122} = h_{135} = h_{143} = h_{157} = h_{164} = h_{186} = h_{1,10,8} = 1$$
$$h_{071} = h_{092} = 1 \tag{1.32}$$
$$h_{211} = h_{222} = h_{234} = h_{246} = h_{253} = h_{268} = h_{275} = h_{2,8,10} = 1$$

all the other elements being zeros.

We also note that, for the original model (1.9)–(1.13) of TR motion, the vector of the voltages u_a supplied to the armature circuits of the DC motors is related, as follows from Eqs (1.19) and (1.23), to the vector of controls u (1.24) of system (1.25)–(1.28) by non-linear transformations of the form (1.19), (6.42)

$$u_a = \Phi_{05}(\hat{z}_2^4, \hat{u}_a) = \Phi_{05}(\Psi_{02}^4(\bar{z}_2^4), \hat{u}_a) = \Phi_{05}(\Psi_{02}^4(\bar{z}_2^4), \hat{U}_a(t_0, \tilde{u}_a(t_0), t, u)) \tag{1.33}$$

where

$$\Psi_{02}^4(\bar{z}_2^4) = \mathrm{col}(\Psi_{02}(\bar{z}_2), \Psi_{03}(\bar{z}_2^3), \Psi_{04}^4(\bar{z}_2^4))$$
$$\hat{u}_a = \mathrm{col}(\hat{u}_{a1}, \hat{u}_{a2}) = \mathrm{col}(\hat{u}_{a1}, u_2) = \mathrm{col}(\tilde{\hat{U}}_{a1}(t_0, \tilde{u}_a(t_0), t, u_1), u_2) \equiv$$
$$\equiv \hat{U}_a(t_0, \tilde{u}_a(t_0), t, u) = \mathrm{col}(\tilde{\hat{U}}_{a1}(t_0, \tilde{u}_a(t_0), t, u_1), \tilde{\hat{U}}_{a2}(u_2)) \tag{1.34}$$
$$\Big(\hat{u}_{a1} = \hat{u}_{a1}(t) = h_1^* \tilde{u}_a(t) = \|1, t - t_0\| \tilde{u}_0(t_0) + \int_{t_0}^{t} (t-s) u_1(s) ds =$$
$$= \hat{u}_a(t_0) + (t - t_0)\tilde{u}_a(t_0) + \int_{t_0}^{t} (t-s) u_1(s) ds \equiv \tilde{\hat{U}}_{a1}(t_0, \tilde{u}_a(t_0), t, u_1)$$
$$\hat{u}_{a2} = \hat{U}_{a2}(u_2) = u_2\Big)$$

$h_1 = \mathrm{col}(1, 0)$ is a two-dimensional vector, the asterisk denotes transposition, and $\tilde{u}_a$ is the state vector (1.31) of the linear system of ODEs

$$\dot{\tilde{u}}_a = P_0 \tilde{u}_a + Q_0 u_1, \quad \tilde{u}_{a0} = \tilde{u}_a(t_0), \quad t \geq t_0 \tag{1.35}$$

where

$$P_0 = \begin{Vmatrix} 0 & 1 \\ 0 & 0 \end{Vmatrix}, \quad Q_0 = \begin{Vmatrix} 0 \\ 1 \end{Vmatrix} \tag{1.36}$$

such that

$$\tilde{u}_a(t) = \mathrm{col}(\tilde{u}_{a1}(t), \tilde{u}_{a2}(t)) = e^{P_0(t-t_0)} \tilde{u}_a(t_0) + \int_{t_0}^{t} e^{P_0(t-s)} Q_0 u_1(s) ds =$$
$$= \begin{Vmatrix} 1 & t - t_0 \\ 0 & 1 \end{Vmatrix} \tilde{u}_a(t_0) + \int_{t_0}^{t} \begin{Vmatrix} t - s \\ 1 \end{Vmatrix} u_1(s) ds \equiv \tilde{U}_a(t_0, \tilde{u}_a(t_0), t, u_1) =$$
$$= \mathrm{col}(\tilde{\hat{U}}_{a1}(t_0, \tilde{u}_a(t_0), t, u_1), \tilde{\hat{U}}_{a2}(t_0, \tilde{u}_a(t_0), t, u_1)), \quad t \geq t_0 \tag{1.37}$$

where $\tilde{U}_{a1}$ is the function defined in (1.34)

$$\tilde{U}_{a2}(t_0, \tilde{u}_a(t_0), t, u_1) = \tilde{u}_{a2}(t) = h_2^* \tilde{u}_a(t) = \tilde{u}_{a2}(t_0) + \int_{t_0}^{t} u_1(s)ds$$

and $h_2 = \mathrm{col}(0, 1)$ is a two-dimensional vector.

2. In what follows the problem will be formulated for equations of the sp-model of TR motion (1.25)–(1.28), (1.24), which is more convenient for preliminary investigation. The problem may be examined in a similar way for the original equations of the model of TR motion (1.9)–(1.13), (1.8).

System (1.25)–(1.28), (1.24) is said to be controllable [6] if, for any two states $z_{p0} \in R^{10}$ and $z_{p1} \in R^{10}$ (where R^n is Euclidean n-space) and any $t_0 < t_1$, $t_1 - t_0 < \infty$, a control $u = u(t)$ (1.24) exists such that the corresponding solution $z(t)$ (1.26) of system (1.25)–(1.28), (1.24) satisfies the boundary conditions

$$z(t_0) = z_{p0}, \quad z(t_1) = z_{p1} \tag{1.38}$$

A solution

$$z = z_p(t), \quad t \in [t_0, t_1] \tag{1.39}$$

of system (1.25)–(1.28), (1.24) satisfying the boundary conditions (1.38) will be called a programmed motion (PM) and the corresponding control

$$u = u_p(t), \quad t \in [t_0, t_1] \tag{1.40}$$

will be called a programmed control.

Let us consider some PM $z_p(t)$ (1.39), (1.38) of system (1.25)–(1.28), (1.24). We shall say that it is stabilizable if a control law exists with feedback with respect to the state vector z,

$$u = u(t, z), \quad t \ge t_0 \tag{1.41}$$

which guarantees asymptotic stability of the PM $z_p(t)$ (1.39), (1.38), in such a way that, after a given time $T_p > 0$ $(t_0 < t_0 + T_p = t_p < t_1)$ (the attenuation time of the transient $e(t) = z(t) - z_p(t)$ in the closed-loop system (1.25)–(1.28), (1.24), (1.41), characterizing the speed of response of the control system), one is guaranteed a prescribed accuracy $\varepsilon_e > 0$ of stabilization of the PM $z_p(t)$ (1.39), (1.38), that is, in such a way as to guarantee satisfaction of the estimate

$$|e(t)| \le \varepsilon_e, \quad \forall t \ge t_p = t_0 + T_p, \quad t_0 < t_p = t_0 + T_p < t_1 \tag{1.42}$$

where everywhere $|a| = (a_1^2 + \dots + a_n^2)^{1/2}$ is the Euclidean norm (magnitude) of the vector $a = \mathrm{col}(a_1, \dots, a_n) \in R^n$.

2. THE EQUATIONS OF TR MOTION IN CANONICAL FORM

The methods proposed below to investigate the controllability conditions for TR, the algorithms for constructing programmed controls and PMs, the synthesis of stabilizing control laws, and the analysis of the stability of PMs of TRs are based on reducing the equations of the sp-model of TR motion (1.25)–(1.28), (1.24), and the equations of the original model of TR motion (1.9)–(1.13), to canonical form by non-linear transformations of the coordinates of the state and control space.

We shall say that the equations of TR motion are in canonical form if they are represented as a linear ODE

$$\dot{x} = Px + Qw, \quad x(t_0) = x_0, \quad t \ge t_0 \tag{2.1}$$

where

$$\begin{gathered} x = \mathrm{col}(x_1, \dots, x_5) = \mathrm{col}(x_1, \dot{x}_1, \dots, x_1^{(4)}) \\ (x_1 = \mathrm{col}(x_B, y_B), x_i = \dot{x}_{i-1}, i = 2, \dots, 5) \end{gathered} \tag{2.2}$$

and x_0 are ten-dimensional vectors of canonical state variables of the TR at the actual and initial instants of time, x_i is a two-dimensional vector, $x_1^{(i)} = x_1^{(i)}(t)$ is the ith derivative with respect to t of $x_1 = x_1(t)$; $x_1^{(0)} = x_1$; $\dot{x}_i = \dot{x}_1^{(i)}$

$$w = \mathrm{col}(w_1, w_2) \tag{2.3}$$

is the two-dimensional vector 'canonical' controls; and P and Q are constant partitioned matrices of dimensions 10×12 and 10×2, of the form

$$P = \begin{Vmatrix} O & I_8 \\ O & O \end{Vmatrix}, \quad Q = \begin{Vmatrix} O \\ I_2 \end{Vmatrix} \tag{2.4}$$

where I_m is the $m \times m$ identity matrix and O is the zero matrix of appropriate order.

3. REDUCTION OF THE EQUATIONS OF THE SP-MODEL OF TR MOTION TO CANONICAL FORM

We shall construct transformations of the coordinates of the space of states z and controls u of the equations of the sp-model of TR motion (1.25)–(1.28), (1.24), which will reduce them to the simpler, canonical form (2.1)–(2.4). We shall look for transformations in the form

$$x = \Psi(z) \tag{3.1}$$

$$w = \Psi_6(z_2^5, u) \tag{3.2}$$

where Ψ and Ψ_6 are ten- and two-dimensional vector functions

$$\Psi(z) = \mathrm{col}(\Psi_1(z_1), \Psi_2(z_2), \Psi_3(z_2^3), \Psi_4(z_2^4), \Psi_5(z_2^5)) \tag{3.3}$$

$$x_1 = \Psi_1(z_1) = z_1 \tag{3.4}$$

Ψ_i $(i = 2, \dots, 6)$ are as yet undetermined two-dimensional vector functions.

We shall describe an algorithm for finding the unknown vector functions Ψ_i $(i = 2, \dots, 6)$. To that end, let us consider the identities

$$\dot{x}_1 = \dot{\Psi}_1(z_1) = \dot{z}_1, \quad x_1^{(i)} = \dot{x}_i = \dot{\Psi}_i(z_2^i) = \sum_{k=2}^{i} \frac{\partial \Psi_i(z_2^i)}{\partial z_k} \dot{z}_k, \quad i = 2, 3, 4, 5 \tag{3.5}$$

where $z_2^i = \mathrm{col}(z_2, \dots, z_i)$, $i \geqslant 2$, $\partial \Psi_i(z_2^i)/(\partial z_k)$ is the 2×2 Jacobian, Substituting into (3.5) the derivatives with respect to time $\dot{x}_i$ $(i = 1, \dots, 5)$ along trajectories of system (2.1)–(2.4), and the derivatives $\dot{z}_i$ $(i = 1, \dots, 5)$ along trajectories of system (1.25)–(1.28), (1.24), we obtain the following relations

$$x_2 = F_1(z_2) = \mathrm{col}(z_{21} \cos z_{22}, z_{21} \sin z_{22}) \equiv \Psi_2(z_2) \tag{3.6}$$

$$x_3 = \frac{\partial \Psi_2(z_2)}{\partial z_2} F_2(z_{21}, z_3) = L_2(z_2) F_2(z_{21}, z_3) = L_3(z_2) z_3 \equiv \Psi_3(z_2^3)$$
$$(x_2 = \dot{x}_1, x_3 = \dot{x}_2 = x_1^{(2)}) \tag{3.7}$$

$$L_2(z_2) = \frac{\partial \Psi_2(z_2)}{\partial z_2} = \frac{\partial F_1(z_2)}{\partial z_2} = \begin{Vmatrix} \cos z_{22} & -z_{21} \sin z_{22} \\ \sin z_{22} & z_{21} \cos z_{22} \end{Vmatrix} \tag{3.8}$$

$$L_3(z_2) = L_2(z_2) D_2(z_{21}) = \begin{Vmatrix} \cos z_{22} & -z_{21}^2 \sin z_{22} \\ \sin z_{22} & z_{21}^2 \cos z_{22} \end{Vmatrix} \tag{3.9}$$

$$x_4 = \frac{\partial\Psi_3(z_2^3)}{\partial z_2}F_2(z_{21}, z_3) + \frac{\partial\Psi_3(z_2^3)}{\partial z_3}F_3(z_4) = K_4(z_2^3) + L_4(z_2)z_4 \equiv \Psi_4(z_2^4) \tag{3.10}$$

$(x_4 = \dot{x}_3 = x_1^{(3)})$

$$K_4(z_2^3) = \frac{\partial\Psi_3(z_2^3)}{\partial z_2}F_2(z_{21}, z_3), \quad L_4(z_2) = \frac{\partial\Psi_3(z_2^3)}{\partial z_3} = L_3(z_2) \tag{3.11}$$

$$x_5 = \frac{\partial\Psi_4(z_2^4)}{\partial z_2}F_2(z_{21}, z_3) + \sum_{k=3}^{4}\frac{\partial\Psi_4(z_2^4)}{\partial z_k}F_k(z_{k+1}) = K_5(z_2^4) + L_5(z_2)z_5 \equiv \Psi_5(z_2^5) \tag{3.12}$$

$(z_5 = \dot{x}_4 = x_1^{(4)})$

$$\begin{aligned} K_5(z_2^4) &= \frac{\partial\Psi_4(z_2^4)}{\partial z_2}F_2(z_{21}, z_3) + \frac{\partial\Psi_4(z_2^4)}{\partial z_3}F_3(z_4) \\ L_5(z_2) &= \frac{\partial\Psi_4(z_2^4)}{\partial z_4} = L_4(z_2) = L_3(z_2) \end{aligned} \tag{3.13}$$

$$\begin{aligned} w &= \frac{\partial\Psi_5(z_2^5)}{\partial z_2}F_2(z_{21}, z_3) + \sum_{k=3}^{4}\frac{\partial\Psi_5(z_2^5)}{\partial z_k}F_k(z_{k+1}) + \\ &+ \frac{\partial\Psi_5(z_2^5)}{\partial z_5}F_5(u) = K_6(z_2^5) + L_6(z_2)u \equiv \Psi_6(z_2^5, u) \end{aligned} \tag{3.14}$$

$(w = \dot{x}_5 = x_1^{(5)})$

$$\begin{aligned} K_6(z_2^5) &= \frac{\partial\Psi_5(z_2^5)}{\partial z_2}F_2(z_{21}, z_3) + \sum_{k=3}^{4}\frac{\partial\Psi_5(z_2^5)}{\partial z_k}F_k(z_{k+1}) \\ L_6(z_2) &= \frac{\partial\Psi_5(z_2^5)}{\partial z_5} = L_5(z_2) = L_3(z_2) \end{aligned} \tag{3.15}$$

Thus, we have constructed the original transformations (3.1) and (3.2) in analytical form (3.1), (3.3), (3.4), (3.6)–(3.13) and (3.14), (3.15), respectively.

We shall show that the original transformations (3.1), (3.3), (3.4), (3.6)–(3.13) and (3.14), (3.15) that we have constructed are uniquely solvable for z and u, respectively. By (3.4), we have

$$z_1 = \Phi_1(x_1) = x_1 \tag{3.16}$$

Let us evaluate the principal minors Δ_1 and Δ_2 of the matrix L_2 (3.8)

$$\begin{aligned} \Delta_1 &= \cos z_{22} > 0 \quad \text{for} \quad z_{22} \in \Omega_{z22} = (-\pi/2, \pi/2) \\ \Delta_2 &= z_{21} \neq 0 \quad \text{for} \quad z_{21} \in \Omega_{z21} = \begin{cases} \Omega_{z21}^+, & \text{if } z_{21} = V_B > 0 \\ \Omega_{z21}^-, & \text{if } z_{21} = V_B < 0 \end{cases} \end{aligned} \tag{3.17}$$

where

$$\Omega_{z21}^+ \equiv (\varepsilon_V, k_V) \tag{3.18}$$

$$\Omega_{z21}^- \equiv (-k_V, -\varepsilon_V) \tag{3.19}$$

are sets, ε_V and k_V being certain positive real numbers, $0 < \varepsilon_V < k_V < \infty$.

Throughout what follows, to fix our ideas (in order to avoid superfluous notation and repeated arguments), we shall consider the case in which the set Ω_{z21}, occurring in (3.17), is of the form (3.18), that is

$$\Omega_{z21} = \Omega^{+}_{z21} \equiv (\varepsilon_V, k_V) \tag{3.20}$$

and introduce a certain parameter $\rho_V = 1$ corresponding to that case.

Note that the case in which the set Ω_{z21}, occurring in Eqs (3.17) is of the form (3.19), that is,

$$\Omega_{z21} = \Omega^{-}_{z21} \equiv (-k_V, -\varepsilon_V) \tag{3.21}$$

may be treated in an entirely analogues fashion by simply replacing the set (3.18) by the set (3.19), the set (3.20) by the set (3.21), and $\rho_V = 1$ by $\rho_V = -1$ everywhere below in Sections 3–5. These will yield estimates and propositions analogous to those derived below.

Thus, in the case when the set Ω_{z21}, occurring in (3.17), is of the form (3.20), it will follow from Theorem 20.9 of [7, p. 484] that the transformation (3.6) is uniquely solvable for z_2 in the rectangular domain

$$\Omega_{\Psi_2} = \{z_2 = \mathrm{col}(z_{21}, z_{22}) \in R^2 : z_{21} \in \Omega_{z21} \equiv \Omega^{+}_{z21}, z_{22} \in \Omega_{z22}\} \tag{3.22}$$

that is, the following inverse transformation exists

$$z_2 = \Phi_2(x_2) \tag{3.23}$$

$$\Phi_2(x_2) = \mathrm{col}(\Phi_{21}(x_2), \Phi_{22}(x_2)) \tag{3.24}$$

$$\Phi_{21}(x_2) = \rho_V(x_{21}^2 + x_{22}^2)^{1/2} \equiv z_{21} = V_B \in \Omega_{z21} \equiv \Omega^{+}_{z21}, \quad \rho_V = 1, \quad x_2 \in \Omega_{\Phi_2} \tag{3.25}$$

$$\Phi_{22}(x_2) = \arcsin(x_{22}/[\Phi_{21}(x_2)]) \in \Omega_{z22}, \quad x_2 \in \Omega_{\Phi_2} \tag{3.26}$$

$$\Omega_{\Phi_2} = \{x_2 = \mathrm{col}(x_{21}, x_{22}) \in R^2 : z_2 = \Phi_2(x_2) \in \Omega_{\Phi_2}\} \tag{3.27}$$

Furthermore, since the matrices L_2 (3.8), L_3 (3.9), L_4 (3.11) and L_5 (3.13) are such that $|\det L_2(z_2)| = |z_{21}| > \varepsilon_V > 0$, $|\det L_i(z_2)| = z_{21}^2 > \varepsilon_V^2 > 0$ $(i = 3, 4, 5, 6)$ for $z_2 \in \Omega_{\Psi_2}$, it follows that

$$\mathrm{rank} L_i(z_2) = 2, \quad z_2 \in \Omega_{\Psi_2}, \quad i = 2, \ldots, 6 \tag{3.28}$$

and inverse matrices $L_i^{-1}(z_2)$ $(i = 2, \ldots, 6)$ exist for the values $z_2 \in \Omega_{\Psi_2}$. Consequently, the transformations (3.7), (3.10), (3.12) and (3.14) are uniquely solvable for z_3, z_4, z_5 and u, respectively, that is, the following inverse transformations exist

$$z_i = \Phi_i(x_2^i), \quad i = 3, 4, 5 \tag{3.29}$$

where

$$\Phi_3(x_2^3) = N_3(x_2)x_3, \quad \Phi_i(x_2^i) = M_i(x_2^{i-1}) + N_i(x_2)x_i, \quad i = 4, 5 \tag{3.30}$$

$$\begin{aligned} M_i(x_2^{i-1}) &= -N_i(x_2)K_i(\Phi_2^{i-1}(x_2^{i-1})), \quad i = 4, 5 \\ N_i(x_2) &= L_i^{-1}(\Phi_2(x_2)) = L_3^{-1}(\Phi_2(x_2)) = N_3(x_2), \quad i = 4, 5 \end{aligned} \tag{3.31}$$

$$\begin{aligned} L_2^{-1}(\Phi_2(x_2)) &= L_{2x}(x_2) = \|l_{2xij}(x_2)\|_{i,j=1,2}, \quad l_{2x1j}(x_2) = x_{2j}/\Phi_{21}(x_2) \\ l_{2x2j}(x_2) &= (-1)^j x_{2,3-j}/[\Phi_{21}(x_2)]^2, \quad j = 1, 2 \end{aligned} \tag{3.32}$$

$$\begin{aligned} \Phi_2^{i-1}(x_2^{i-1}) &= \mathrm{col}(\Phi_2(x_2), \Phi_3(x_2^3), \ldots, \Phi_{i-1}(x_2^{i-1})) \\ u &= \Phi_6(x_2^5, w) \end{aligned} \tag{3.33}$$

$$\Phi_6(x_2^5, w) = M_6(x_2^5) + N_6(x_2)w \tag{3.34}$$

$$M_6(x_2^5) = -N_6(x_2)K_6(\Phi_2^5(x_2^5)), \quad N_6(x_2) = L_6^{-1}(\Phi_2(x_2)) \tag{3.35}$$

Thus, taking relations (3.16), (3.23)–(3.32) into consideration, we have constructed a one-to-one inverse transformation for the original transformation (3.1), (3.3), (3.4), (3.6)–(3.11)

$$z = \Phi(x), \quad x \in \Omega_\Phi \tag{3.36}$$

where

$$\Phi(x) = \mathrm{col}(\Phi_1(x_1), \Phi_2(x_2), \Phi_3(x_2^3), \Phi_4(x_2^4), \Phi_5(x_2^5)) \tag{3.37}$$

Φ_i $(i = 1, \ldots, 5)$ are the vector functions (3.16), (3.23)–(3.32), and

$$\Omega_\Phi = \{x = \mathrm{col}(x_1, \ldots, x_5) \in R^{10} : z = F(x) \in \Omega_\Psi\} \tag{3.38}$$

$$\Omega_\Psi = \{z = \mathrm{col}(z_1, \ldots, z_5) \in R^{10} : z_i \in R^2, i = 1, 3, 4, 5; z_2 \in \Omega_{\Psi_2}\} \tag{3.39}$$

We shall now establish the following fact. Take any solution $x_1(t)$ of the ODE

$$x_1^{(5)} = \Psi_6(\Phi_2^5(\dot{x}_1, x_1^{(2)}, x_1^{(3)}, x_1^{(4)}), u) \tag{3.40}$$

which is equivalent to system (2.1)–(2.4) for $w = \Psi_6(\Phi_2^5(\dot{x}_1, x_1^{(2)}, x_1^{(3)}, x_1^{(4)}), u)$, where $x_2^5 = \mathrm{col}(x_2, \ldots, x_5) = \mathrm{col}(\dot{x}_1, x_1^{(2)}, x_1^{(3)}, x_1^{(4)})$, substitute it into system (3.5)

$$x_1^{(i)} = \dot{x}_i = \dot{\Psi}_i(z_2^i) = \Psi_{i+1}(z_2^{i+1}) = x_{i+1}, \quad i = 1, 2, 3, 4 \tag{3.41}$$

where $z_2^1 = z_2$, and use this system to define vector functions $z_i(t)$, $(i = 2, 3, 4, 5)$. Then the system of vector functions

$$x_1(t) = z_1(t), z_2(t), z_3(t), z_4(t), z_5(t) \tag{3.42}$$

will be a solution of system (1.25)–(1.28), (1.24).

Let us substitute the system of vector functions (3.42) into system (1.25)–(1.28), (1.24), thereby converting all the equations of that system into identities, in particular, obtaining the identity

$$\dot{x}_1 = \dot{z}_1 \equiv F_1(z_2) \tag{3.43}$$

Differentiating this identity with respect to t, we obtain

$$\ddot{x}_1 = \ddot{z}_1 = \dot{x}_2 = \dot{\Psi}_2(z_2) = \frac{\partial \Psi_2(z_2)}{\partial z_2}\dot{z}_2 \tag{3.44}$$

For the moment, it is not yet possible to replace $\dot{z}_2$ by the vector function F_2, because we have yet to show that the vector functions $x_1(t)$, $z_2(t)$, $\ldots$, $z_5(t)$, obtained as described above from Eq. (3.40) and system (3.41), satisfy the sp-system of ODEs (1.25)–(1.28), (1.24) – that is precisely what we have to prove.

Subtracting identity (3.7) term by term from identity (3.44), we obtain

$$\frac{\partial \Psi_2(z_2)}{\partial z_2}(\dot{z}_2 - F_2(z_{21}, z_3)) \equiv 0 \tag{3.45}$$

Similarly, differentiating the identities $x_i = \Psi_i(z_2^i)$ $(i = 3, 4, 5)$ (3.41) with respect to t

$$\dot{x}_i = \dot{\Psi}_i(z_2^i) = \sum_{k=2}^{i} \frac{\partial \Psi_i(z_2^i)}{\partial z_k}\dot{z}_k, \quad i = 3, 4, 5$$

and subtracting the respective identities

$$\dot{x}_i = \dot{\Psi}_i(z_2^i) = \frac{\partial \Psi_i(z_2^i)}{\partial z_2} F_2(z_{21}, z_3) + \sum_{k=3}^{i} \frac{\partial \Psi_i(z_2^i)}{\partial z_k} F_k(z_{k+1}), \quad i = 3, 4, 5; \; z_6 = u$$

from (3.10), (3.12) and (3.14), we obtain

$$\sum_{k=3}^{i} \frac{\partial \Psi_i(z_2^i)}{\partial z_k} (\dot{z}_k - F_k(z_{k+1})) \equiv 0, \quad i = 3, 4, 5 \tag{3.46}$$

Let us write Eqs (3.45) and (3.46) for the unknowns

$$\dot{z}_2 - F_2(z_{21}, z_3), \quad \dot{z}_k - F_k(z_{k+1}), \quad k = 3, 4, 5$$

in the form of a system

$$J_0(z_2^4)(\dot{z}_2^5 - F_2^5(z_{21}, z_3^6)) = 0 \tag{3.47}$$

where

$$z_3^6 = \mathrm{col}(z_3, \ldots, z_6), \quad z_6 = u, \quad F_2^5(z_{21}, z_3^6) = \mathrm{col}(F_2(z_{21}, z_3), F_3(z_4), F_4(z_5), F_5(z_6))$$

and

$$J_0(z_2^4) = \frac{\partial \Psi_0^5(z_2^5)}{\partial z_2^5} \tag{3.48}$$

is the 8×8 Jacobian, $\Psi_2^5(z_2^5) = \mathrm{col}(\Psi_2(z_2), \Psi_3(z_2^3), \ldots, \Psi_5(z_2^5))$.

Taking relations (3.6)–(3.13) into consideration, we conclude that the matrix function J_0 (3.48) is a lower triangular partitioned matrix with 2×2 diagonal blocks L_i ($i = 2, \ldots, 5$) (3.8), (3.9), (3.11), (3.13) which, according to condition (3.28), are non-singular. Therefore

$$\mathrm{rank} J_0(z_2^4) = 8, \quad \forall z_2^4 \in \Omega_{J0} \tag{3.49}$$

where

$$\Omega_{J0} = \{z_2^4 = \mathrm{col}(z_2, z_3, z_4) \in R^6 : z_2 \in \Omega_{\Psi_2}, z_3, z_4 \in R^2\} \tag{3.50}$$

Consequently, taking (3.49) into consideration, we conclude that the matrix function J_0 (3.48) is also non-singular. Hence it follows that, at each point of the set Ω_{J0} (3.50), system (3.47) has only the trivial solution

$$\dot{z}_2^5 - F_2^5(z_{21}, z_3^6) = 0$$

Noting also the identity (3.43), we conclude that the vector function $y = \mathrm{col}(x_1, z_2 \ldots, z_5) \equiv z$ is a solution of the sp-system of ODEs (1.25)–(1.28), (1.24).

4. CONTROLLABILITY AND ALGORITHMS FOR CONSTRUCTING PROGRAMMED CONTROLS AND MOTIONS OF THE TR

We shall first show that the model of TR motion in canonical form (2.1)–(2.4) is completely controllable [8, p. 269]. Since the matrix

$$S = \| Q, PQ, \ldots, P^9 Q \| \tag{4.1}$$

has a submatrix $S_0 = \| Q, PQ, \ldots, P^4 Q \|$ for which, by (2.4), $|\det S_0| = 1$, and so

$$\mathrm{rank} S = \mathrm{rank} S_0 = 10 \tag{4.2}$$

it follows that system (2.1)–(2.4) is completely controllable [8, p. 269, Theorem 3.1], that is, a control law

$$w = w_p = w_p(t) = Q^* e^{P^*(t_1 - t)} K_0^{-1}(x_{p1} - e^{PT} x_{p0}) \tag{4.3}$$

exists, where

$$K_0 = \int_{t_0}^{t_1} e^{P(t_1 - t)} Q Q^* e^{P^*(t_1 - t)} dt \tag{4.4}$$

is a constant positive-definite 10×10 matrix (by virtue of the complete controllability of system (2.1)–(2.4) [8]) taking system (2.1)–(2.4) from any initial state $x_p(t_0) = x_{p0} = \Psi(z_{p0}) \in R^{10}$ (in particular, for $z_{p0} \in \Omega_\Psi$, where Ω_Ψ is the set (3.39)) to an arbitrary terminal state $x_p(t_1) = x_{p1} = \Psi(z_{p1}) \in R^{10}$ (in particular, for $z_{p1} \in \Omega_\Psi$) in a time $t_1 - t_0 < \infty$ along the trajectory

$$x_p = x_p(t) = e^{P(t - t_0)} x_{p0} + \int_{t_0}^{t} e^{P(t - s)} Q w_p(s) ds, \quad t \in [t_0, t_1] \tag{4.5}$$

Note that in order to evaluate $e^{P(t - t_0)}$, $e^{P(t - s)}$, $e^{P(t_1 - t)}$, e^{PT}, where P is the matrix (2.4), we can use the representation of $e^{P\tau}$ as

$$e^{P\tau} = \sum_{i=0}^{4} \frac{P^i \tau^i}{i!}$$

Hence, using transformations (3.6)–(3.13), we deduce that the control law

$$\begin{aligned} u = u_p &= \mathrm{col}(u_{p1}, u_{p2}) = \Phi_6(\overset{5}{x}_{p2}, w_p) = \Phi_6(\Psi_2^5(\overset{5}{z}_{p2}), w_p) = \\ &= \mathrm{col}(\Phi_{61}(\Psi_2^5(\overset{5}{z}_{p2}), w_p), \Phi_{62}(\Psi_2^5(\overset{5}{z}_{p2}), w_p)) \\ &(u_{pi} = \Phi_{6i}(\Psi_2^5(\overset{5}{z}_{p2}), w_p), \; i = 1, 2) \end{aligned} \tag{4.6}$$

where

$$\Psi_2^5(\overset{5}{z}_{p2}) = \mathrm{col}(\Psi_2(z_{p2}), \Psi_3(\overset{3}{z}_{p2}), \ldots, \Psi_5(\overset{5}{z}_{p2}))$$

and w_p and x_p are defined as in (4.3)–(3.4), will steer the sp-model of TR motion (1.25)–(1.28), (1.24) from any initial state $z_p(t_0) = z_{p0} \in \Omega_\Psi$, to an arbitrary terminal state $z_p(t_1) = z_{p1} \in \Omega_\Psi$, where Ω_Ψ is the set (3.39), in a time $t_1 - t_0 < \infty$ along the trajectory

$$z = z_p = \Phi(x_p, t) \quad t \in [t_0, t_1] \tag{4.7}$$

Therefore, the sp-model of TR motion (1.25)–(1.28), (1.24) is also controllable.

We shall now show that the original model of TR motion (1.9)–(1.13), (1.8) is also controllable.

Using transformation (1.33), we deduce that the control law

$$u_a = u_{ap} = \Phi_{05}(\overset{4}{\hat{z}}_{p2}, \hat{u}_{ap}) = \Phi_{05}(\Psi_{02}^4(\overset{4}{\hat{z}}_{p2}), \hat{u}_{ap}) \tag{4.8}$$

where, by (1.34),

$$\begin{aligned} \hat{u}_{ap} &= \mathrm{col}(\hat{u}_{ap1}, \hat{u}_{ap2}) = \hat{U}_a(t_0, \tilde{\hat{u}}_{ap}(t_0), t, u_p) = \\ &= \mathrm{col}(\tilde{\hat{U}}_{ap1}(t_0, \tilde{\hat{u}}_{ap}(t_0), t, u_{p1}), \tilde{\hat{U}}_{ap2}(u_{p2})) \\ &(\hat{u}_{ap1} = h_1^* \tilde{\hat{u}}_{ap} = \tilde{\hat{u}}_{ap1} = \tilde{\hat{u}}_{ap1}(t) = \hat{u}_{ap1}(t_0) + (t - t_0)\tilde{\hat{u}}_{ap2}(t_0) + \\ &+ \int_{t_0}^{t} (t - s) u_{p1}(s) ds \equiv \tilde{\hat{U}}_{ap1}(t_0, \tilde{\hat{u}}_{ap}(t_0), t, u_{p1}) \\ &\hat{u}_{ap2} \equiv \hat{U}_{ap2}(u_{p2}) = u_{p2}) \end{aligned} \tag{4.9}$$

u_{pi} $(i = 1, 2)$ are the vectors defined by (4.6), (4.3), (4.4), (4.7) and (4.5), takes the initial model of TR motion (1.9)–(1.13) from any initial state $\bar{z}_p(t_0) = \bar{z}_{p0} \in \Omega_{\Psi 0}$ to an arbitrary terminal state $\bar{z}_p(t_1) = \bar{z}_{p1} \in \Omega_{\Psi 0}$, where $\Omega_{\Psi 0}$ is the set (6.39), (6.9), in a time $t_1 - t_0 < \infty$ along the trajectory

$$\bar{z}_p = \Phi_0(H_2 z_p), \quad z_p \in \Omega_\Psi, \quad t \in [t_0, t_1] \tag{4.10}$$

Therefore, the original model of TR motion (1.9)–(1.13), (1.8) is also controllable.

5. CRITERIA FOR THE STABILIZABILITY OF PMs OF A TR

1. We will first consider the problem of synthesizing stabilizing control laws w and analysing the stability of a PM belonging to the set Ω_Φ (3.38), $t \geqslant t_0$, for the canonical model of TR motion (2.1)–(2.4).

It follows from the complete controllability of the model (the validity of Eqs (4.1), (4.2)) [8, p. 274, Theorem 4.1] that a constant 2×10 gain matrix

$$\Gamma_0 = \|\Gamma_{01}, \ldots, \Gamma_{05}\| \tag{5.1}$$

exists, where $\Gamma_{0j}(j = 1, \ldots, 5)$ is a (2×2) partitioned matrix, such that the matrix

$$\Gamma = P + Q\Gamma_0 \tag{5.2}$$

will have given eigenvalues $\lambda_i (i = 1, \ldots, 10)$, in particular, for example, so that the matrix Γ will be stable (Hurwitzian) [8, p. 597], that is, $\mathrm{Re}\lambda_i < 0$ $(i = 1, \ldots, 10)$. Moreover, the matrix Γ_0 (5.1) may be chosen in such a way that the matrix Γ (5.2) will have, say, given distinct real negative eigenvalues, that is

$$\lambda_i < 0 \quad (\lambda_i \neq \lambda_j,\ i \neq j;\ i, j = 1, \ldots, 10) \tag{5.3}$$

Let us synthesize a control law with "canonical" feedback with respect to x, in the form

$$w = w_p + \Gamma_0(x - x_p) \tag{5.4}$$

Then the equation of the transients $e_x = x - x_p$ in the closed-loop system (2.1)–(2.4), (5.4), (5.3) will have the form

$$\dot{e}_x = \Gamma e_x, \quad e_x(t_0) = e_{x0}, \quad t \geq t_0 \tag{5.5}$$

Consequently, the PM $x_p(t)$ (4.5) of system (2.1)–(2.3), (5.4), (5.1)–(5.3) is asymptotically stable in the large with an estimate

$$|e_x(t)| \leq \beta_0 |e_x(t_0)| \exp[\gamma_0(t - t_0)], \quad e_x(t_0) = e_{x0}, \quad t \geq t_0 \tag{5.6}$$

and moreover the transient $e_x(t)$ will attenuate in the given aperiodic manner (in particular, for e_{x0} such that $e_{x0} + x_{p0} = x_0 = x(t_0) \in \Omega_\Phi$). Here

$$\gamma_0 = \max_i \lambda_i, \quad \lambda_i < 0, \quad i = 1, \ldots, 10; \quad \beta_0 = \sum_{i=1}^{10} |\bar{\Gamma}_i| > 0$$

where

$$\bar{\Gamma}_i = \left[\prod_{\substack{k=1\\k \neq i}}^{10} (\Gamma - \lambda_k I_{10})\right]\left[\prod_{\substack{k=1\\k \neq i}}^{10} (\lambda_i - \lambda_k)\right]^{-1}, \quad i = 1, \ldots, 10$$

are the coefficient matrices of the Lagrange–Sylvester interpolation polynomial [9, p. 49]

$$e^{\Gamma(t - t_0)} = \sum_{i=1}^{10} \bar{\Gamma}_i \exp[\lambda_i(t - t_0)]$$

and $|A| = \Sigma_{i=1}^{n}\Sigma_{j=1}^{n} a_{ij}^{2})^{1/2}$ is the Euclidean norm (magnitude) of the $n \times n$ matrix $A = \|a_{ij}\|_{i,j=1,\ldots,n}$.
Suppose

$$\varepsilon_{ex} = \mu_0^{-1}\varepsilon_e > 0 \quad (|e_x(t)| \le \varepsilon_{ex}, \forall t \ge t_p = t_0 + T_p, t_0 < t_p = t_0 + T_p < t_1) \tag{5.7}$$

(where $0 < \mu_0 < +\infty$ is some constant, which will be defined below in (5.27), (5.28); $\varepsilon_e > 0$ is the prescribed precision of the stabilization of the PM $z_p(t)$ (1.39), (1.38) of system (1.25)–(1.28), (1.24), (1.41)) is the prescribed precision of realization of the PM $x_p(t)$ (4.5), and $T_p = t_p - t_0 > 0$ ($t_0 < t_p = t_0 + T_p < t_1$) is the prescribed attenuation time of the transient $e_x(t) = x(t) - x_p(t)$, characterizing the speed of response of the control system. Using the expression

$$\varepsilon_{ex} = \beta_0 |e_x(t_0)| \exp[\gamma_0 T_p] \tag{5.8}$$

one can now obtain an estimate of the form

$$T_p = t_p - t_0 = -\frac{1}{\gamma_0}\ln\frac{\beta_0|e_x(t_0)|}{\varepsilon_{ex}} \tag{5.9}$$

from which one can obtain the relation

$$|\gamma_0| \ge \frac{1}{T_p}\ln\frac{\beta_0|e_x(t_0)|}{\varepsilon_{ex}} \tag{5.10}$$

which can be used, thanks to the presence of the gain matrix Γ_0 (5.1) in the control law w (5.4), to choose the eigenvalues $\lambda_i (i = 1, \ldots, 10)$ (5.3) of the matrix Γ (5.2) in a well-founded manner.

We have thus shown that a PM $x_p(t)$ (4.5) of the canonical model of TR motion (2.1)–(2.4), closed by the control law w (5.4), (5.1)–(5.3), (5.10), is asymptotically stable in the large with an estimate (5.6) for the magnitude $|e_x(t)|$ of the transient $e_x(t) = x(t) - x_p(t)$, so that one can guarantee the prescribed precision ε_{ex} (5.7) and attenuation time $T_p = t_0 - t_p > 0$ ($t_0 < t_p = t_0 + T_p < t_1$) of the transient $e_x(t)$ (that is, the PM $x_p(t)$ is stabilizable in the sense of the definition given in Section 1).

2. When solving the problem of synthesizing stabilizing control laws u and u_a and analysing the stability of a PM $z_p(t)$ of the sp-model of TR motion (1.25)–(1.28), (1.24) and of a PM $\tilde{z}_p(t)$ of the original model of TR motion (1.9)–(1.13), (1.8) we will use the following.

Lemma. Suppose the following conditions are satisfied:
(1) the system of ODEs

$$\dot{e} = \bar{F}_e(e, t), \quad e(t_0) = e_0, \quad t \ge t_0 \tag{5.11}$$

(where F is a vector function defined on the set

$$\Omega_{\bar{F}_e} = \{(e, t) : e \in \Omega_e \subset R^n, t \in [t_0, +\infty)\}$$

which satisfies the conditions for the existence and uniqueness of a solution of system (5.11), and moreover $\bar{F}_e(0, t) \equiv 0$), subjected to the continuously differentiable transformation

$$e = \Delta\Phi(e_x, t) \tag{5.12}$$

(where the vector function $\Delta\Phi(e_x, t)$ is defined on the set

$$\Omega_{\Delta\Phi} = \{(e_x, t) : e_x \in \Omega_{ex} \subset R^n, t \in [t_0, +\infty)\} \tag{5.13}$$

and moreover

$$|\Delta\Phi(e_x, t)| \le |\Delta\tilde{\Phi}(e_x)|, \quad (e_x, t) \in \Omega_{\Delta\Phi}, \quad \Delta\Phi(0, t) \equiv 0 \tag{5.14}$$

with $\Delta\tilde{\Phi}(e_x)$ a continuous vector function, $\Delta\tilde{\Phi}(0) \equiv 0$), is transformed to an equation

$$\dot{e}_x = \bar{F}_{ex}(e_x, t), \quad e_x(t_0) = e_{x0}, \quad t \ge t_0 \tag{5.15}$$

(where the vector function $\bar{F}_{ex}$ is defined on the set

$$\Omega_{\bar{F}ex} = \{(e_x, t) : (e_x, t) \in \Omega_{\Delta\Phi}, e = \Delta\Phi(e_x, t) \subset \Omega_{\bar{F}e}\}$$

and moreover $\bar{F}_{ex}(0, t) \equiv 0$)

(2) the transformation (5.12) is uniquely solvable for e_x, that is, there is a unique inverse transformation

$$e_x = \Delta\Phi^{-1}(e, t) \equiv \Delta\Psi(e, t) \tag{5.16}$$

where the continuously differentiable vector function $\Delta\Psi(e, t)$ is defined on the set

$$\Omega_{\Delta\Psi} = \{(e, t) : e = \Delta\Phi(e_x, t) \subset \Omega_{\bar{F}_e}, (e_x, t) \in \Omega_{\Delta\Phi}\} \tag{5.17}$$

and moreover

$$|\Delta\Psi(e, t)| \le |\Delta\tilde{\Psi}(e)|, \quad (e, t) \in \Omega_{\Delta\Psi}, \quad \Delta\Psi(0, t) \equiv 0 \tag{5.18}$$

$\Delta\tilde{\Psi}(e)$ being a continuous vector function, $\Delta\tilde{\Psi}(e) \equiv 0$.

Under the conditions listed above, the solution $e = 0$ of Eq. (5.11) is asymptotically Lyapunov stable if and only if the corresponding solution $e_x = 0$ of Eq. (5.15) is asymptotically Lyapunov stable.

Proof. We first show that asymptotic Lyapunov stability of the solution $e_x = 0$ of system (5.15) implies the same property of the solution $e = 0$ of system (5.11).

To that end, we shall first show that the Lyapunov stability of the solution $e_x = 0$ of system (5.15) implies that of the solution $e = 0$ of system (5.11).

Take any $\varepsilon > 0$. Since (by the first assumption of the lemma) it follows from the estimate (5.14) and the continuity of the vector function $\Delta\Phi(e_x)$ that the vector function $\Delta\Phi(e_x, t)$ is continuous with respect to e_x at $e_x = 0$, uniformly with respect to $t \in [t_0, \infty)$, if follows that, for the given $\varepsilon > 0$, we can find $\varepsilon_0 > 0$ such that

$$|e_x| < \varepsilon_0 \Rightarrow |e| = |\Delta\Phi(e_x, t)| \le |\Delta\tilde{\Phi}(e_x)| < \varepsilon, \quad t \in [t_0, \infty) \tag{5.19}$$

where $e_x = e_x(e_{x0}, t)$, $e = e(e_0, t)$.

Furthermore, since the solution $e_x = 0$ of system (5.15) is Lyapunov stable, it follows that, given $\varepsilon_0 > 0$, we can find $\delta_0 > 0$ such that

$$|e_{x0}| < \delta_0 \Rightarrow |e_x(e_{x0}, t)| < \varepsilon_0, \quad t \in [t_0, \infty) \tag{5.20}$$

Consider the vector function $\Delta\Psi(e, t)$. Using the continuity with respect to e at $e = 0$ of the vector function $\Delta\Psi(e, t_0)$, and with $\delta_0 > 0$ as found above, we can find a $\delta > 0$ such that

$$|e_0| < \delta \Rightarrow |e_{x0}| = |\Delta\Psi(e_0, t_0)| < \delta_0 \tag{5.21}$$

By inequalities (5.19)–(5.21), we obtain

$$|e_0| < \delta \Rightarrow |e_{x0}| < \delta_0 \Rightarrow |e_x(e_{x0}, t)| < \varepsilon_0 \Rightarrow |e(e_0, t)| < \varepsilon, \quad t \ge t_0$$

Consequently, Lyapunov stability of the solution $e_x = 0$ of system (5.15) implies that of the solution $e = 0$ system (5.11).

Now, it follows from the asymptotic Lyapunov stability of the solution $e_x = 0$ of system (5.15) and from the continuity of the vector function $\Delta\tilde{\Phi}(e_x)$ and Eq. (5.14) that

$$|e_x(t)| \xrightarrow[t \to \infty]{} 0 \Rightarrow |\Delta\tilde{\Phi}(e_x(t))| \xrightarrow[t \to \infty]{} 0 \Rightarrow |\Delta\Phi(e_x(t), t)| = |e(t)| \xrightarrow[t \to \infty]{} 0$$

Consequently, the solution $e = 0$ of system (5.11) is asymptotically Lyapunov stable.

Similarly (using the second assumption of the lemma, the continuity of the vector function $\Delta\tilde{\Psi}(e_x)$ and Eq. (5.18)), it can be proved that asymptotic Lyapunov stability of the solution $e = 0$ of system (5.11) implies that of the solution $e_x = 0$ of system (5.15). This proves the lemma.

3. We will now synthesize a stabilizing control law u with feedback with respect to z for the sp-model of TR motion (1.25)–(1.28), (1.24).

Let us consider a PM

$$z_p(t) \in \Omega_{zp}, \quad t \geq t_0 \tag{5.22}$$

where

$$\begin{aligned} \Omega_{zp} &= \{z_p = \mathrm{col}(z_{p1}, \ldots, z_{p5}) \in R^{10} : z_{p1} \in R^2, z_{p2} \in \Omega_{\Psi_2}; \\ \sup_{t \geq t_0}|z_{pik}(t)| &= k_{zpik} < \infty \quad (i = 3, 4, 5;\ k = 1, 2)\} \end{aligned} \tag{5.23}$$

in which Ω_{Ψ_2} is the set (3.22) and $k_{zpik} \geqslant 0$ ($i = 3, 4, 5$; $k = 1, 2$) are certain constants.

Substituting relations (5.4), (5.1)–(5.3) and (5.10) into (3.33) and using transformations (3.3)–(3.4), (3.6)–(3.11) of the state space coordinates, we obtain the desired stabilizing control law with feedback with respect to z

$$\begin{aligned} u &= \mathrm{col}(u_1, u_2) = \Phi_6(x_2^5, w_p + \Gamma_0(x - x_p)) = \\ &= \Phi_6(\Psi_2^5(z_2^5), w_p + \Gamma_0(\Psi(z) - \Psi(z_p))) \equiv \bar{\Phi}_6(\Gamma_0, t, z) = \\ &= \mathrm{col}(\bar{\Phi}_{61}(\Gamma_0, t, z), \bar{\Phi}_{62}(\Gamma_0, t, z)) \\ &(u_i = \bar{\Phi}_{6i}(\Gamma_0, t, z) = h_i^* \Phi_6(\Psi_2^5(z_2^5), w_p + \Gamma_0(\Psi(z) - \Psi(z_p))),\ i = 1, 2) \end{aligned} \tag{5.24}$$

where $h_1 = \mathrm{col}(1, 0)$, and $h_2 = \mathrm{col}(0, 1)$ are two-dimensional vectors, for the sp-model of TR motion (1.25)–(1.28), (1.24).

The equation of the transient $e = z - z_p$ in the closed sp-model of TR motion (1.25)–(1.28), (1.24), (5.24), (5.1)–(5.3), (5.10) has the form

$$\dot{e} = F_e(e, t), \quad e(t_0) = e_0, \quad t \geq t_0 \tag{5.25}$$

where

$$F_e(e, t) = F(e + z_p, \Phi_6(\Psi_2^5(e_2^5 + z_{p2}^5), w_p + \Gamma_0(\Psi(e + z_p) - \Psi(z_p)))) - F(z_p, u_p) \tag{5.26}$$

$e_0 + z_{p0} = z_0 \in \Omega_\Psi$, Ω_Ψ is the set (3.39), $F_e(0, t) \equiv 0$.

Let us estimate the transient e in Eqs (5.25) and (5.26). Using the finite-increments formula [10, p. 122, Lemma 3.1] for the vector function $\Delta\Phi(e_x, t) = \Phi(e_x + x_p) - \Phi(x_p)$, relations (5.22), (5.23), (3.38), (3.39), (3.27), the estimate (5.6) for the magnitude $|e_x(t)|$ of the vector $e_x(t)$, and the estimate

$$|A| \leq \sum_{i=1}^{n} \sum_{j=1}^{m} |a_{ij}|$$

for the norm $|A|$ of an $n \times m$ matrix A, we estimate the magnitude $|e|$ of the vector $e = z - z_p$. We obtain

$$\begin{aligned} |e(t)| &= |z(t) - z_p(t)| = |\Phi(e_x + x_p) - \Phi(x_p)| = |\Delta\Phi(e_x, t)| = \\ &= \left|\left[\int_0^1 J_{\Delta\Phi}(se_x(t), t)ds\right] e_x(t)\right| \leq \left|\int_0^1 J_{\Delta\Phi}(se_x(t), t)ds\right| |e_x(t)| \leq \\ &\leq \mu_0 |e_x(t)| \equiv \Delta\tilde{\Phi}_0(e_x(t)) \leq \mu_0\beta_0|e_x(t_0)| \exp[\gamma_0(t - t_0)] = \\ &= \mu|\Delta\Psi(e_0, t_0)| \exp[\gamma_0(t - t_0)], \quad t \geq t_0 \end{aligned} \tag{5.27}$$

where

$$\begin{aligned}
&e(t_0) + z_p(t_0) = z(t_0) = z_0 \in \Omega_\Psi \\
&J_{\Delta\Phi}(e_x, t) = \partial\Delta\Phi(e_x, t)/(\partial e_x) \\
&\sup_{\sigma_x \in [0, e_x],\, t \geq t_0} |J_{\Delta\Phi}(\sigma_x, t)| = \mu_0 < +\infty \\
&[0, e_x] = \{\eta_{ex} : \eta_{ex} = s e_x, e_x + x_p = x \in \Omega_\Phi, 0 \leq s \leq 1\}
\end{aligned} \tag{5.28}$$

$0 < \mu_0 < +\infty$ is some constant, $\mu = \mu_0\beta_0$, $\Delta\tilde{\Phi}_0(0) \equiv 0$.

Relations (5.27) and (5.28) imply the estimate

$$\begin{aligned}
&|\Delta\Phi(e_x(t), t)| \leq \Delta\tilde{\Phi}_0(e_x(t)) \equiv \mu_0|e_x(t)| \\
&e_x(t_0) + x_p(t_0) = x(t_0) = x_0 \in \Omega_\Phi, \quad t \geq t_0
\end{aligned}$$

from which it follows that the vector function $\Delta\Phi(e_x, t)$ is continuous with respect to e_x at $e_x = 0$, uniformly in $t \in [t_0, \infty)$, and moreover $\Delta\Phi(0, t) \equiv 0$; consequently, the first assumption of the lemma is valid.

Hence, using the continuity of the vector function $\Delta\Psi(e, t) = \Psi(e + z_p) - \Psi(z_p)$, where Ψ is the vector function (3.3), (3.4), (3.6)–(3.13), and the fact that, since the solution $e_x = 0$ of system (5.5), (5.2), (5.3) is asymptotically Lyapunov stable, it follows by the above lemma that the solution $e = 0$ of system (5.25), (5.26), (5.2), (5.3) is also asymptotically Lyapunov stable, and relations (5.6)–(5.10), (5.2), (5.3) and (5.1), we deduce that, if the sp-model of TR motion (1.25)–(1.28), (1.24) is closed by the control law u (5.24), (5.1)–(5.3), (5.10), then the prescribed precisions $\varepsilon_e > 0$ of realization of the PM $z_p(t)$ (5.22), (5.23) and attenuation time $T_p = t_p - t_p > 0$ ($t_0 < t_p = t_0 + T_p < t_1$) of the transient $e = z - z_p$ are guaranteed in such a way that the estimate (1.42) is valid.

We have thus proved the following theorem.

Theorem 1. Let $z_p(t)$ (5.22), (5.23) be a given (constructed) PM for the sp-model of TR motion (1.25)–(1.28), (1.24).

Then the stabilizing control law u (5.24), (5.1)–(5.3), (5.10) with feedback with respect to z guarantees asymptotic stability of the PM $z_p(t)$ (5.22), (5.23); the transient $e(t) = z(t) - z_p(t)$ in the closed sp-model of TR motion (1.25)–(1.28), (5.24), (5.1)–(5.3), (5.10) satisfies the estimate (5.27), (5.28) one can guarantee prescribed precision $\varepsilon_e > 0$ of the realization of the PM $z_p(t)$ (5.22), (5.23) and attenuation time $T_p = t_p - t_0 > 0$ ($t_0 < t_p = t_0 + T_p < t_1$) of the transient $e = z - z_p$, in such a way that estimate (1.42) is valid.

4. We will now consider the problem of synthesizing a stabilizing control law u_a and of analysing the stability of a PM

$$\bar{z}_p = \bar{z}_p(t) \in \Omega_{\bar{z}p}, \quad t \geq t_0 \tag{5.29}$$

where

$$\Omega_{\bar{z}p} = \{\bar{z}_p \in R^8 : \bar{z}_p = \Phi_0(H_2 z_p), z_p \in \Omega_{zp}\} \tag{5.30}$$

for the original model of TR motion (1.9)–(1.13), (1.8).

Substituting the control law u (5.24), (5.1)–(5.3), (5.10) into (1.33), (1.34), and noting the estimate (5.27), (5.28) of the absolute value $|e(t)|$ of the solution $e(t) = z(t) - z_p(t)$ of system (5.25), (5.26), which follows from Theorem 1, we obtain a stabilizing law for the variation of the vector of control voltages

$$u_a = \Phi_{05}(\Psi^4_{02}(\bar{z}^4_2), \hat{U}_a(t_0, \tilde{\hat{u}}_a(t_0), t, \bar{\Phi}_6(\Gamma_0, t, z))) \tag{5.31}$$

supplied to the armature circuits of the DC motors, which guarantee asymptotic stability of the PM $\bar{z}_p$ (5.29), (5.30) with an estimate $|\bar{e}(t)|$ for the transient

$$\begin{aligned}
\bar{e} &= \bar{z} - \bar{z}_p = \Phi_0(\hat{e} + \hat{z}_p) - \Phi_0(\hat{z}_p) \equiv \Delta\Phi_0(\hat{e}, t) = \\
&= \Phi_0(H_2(e + z_p)) - \Phi_0(H_2 z_p) \equiv \Delta\Phi_0(e, t)
\end{aligned} \tag{5.32}$$

(indicated below in (5.39) and (5.40)), where

$$\hat{z}_p = \hat{z}_p(t) = H_2 z_p(t), \quad z_p(t) \in \Omega_{zp}, \quad t \geq t_0 \tag{5.33}$$

in the original closed model of TR motion (1.9)–(1.13), (5.31), (5.1)–(5.3), (5.10).

We first note that, since the solution $e = 0$ of system (5.25), (5.26), (5.2), (5.3), (5.10) is asymptotically Lyapunov stable, it follows that the solution $\hat{e} = 0$ ($\hat{e} = H_2 e$ and $\hat{e} = H_2 e = 0$ for $e = 0$) of the system

$$\dot{\hat{e}} = \hat{F}_e(\hat{e}, t), \ \hat{e}(t_0) = \hat{e}_0, \quad \hat{e}_0(t_0) + \hat{z}_p(t_0) = \hat{z}(t_0) = \hat{z}_0 \in \Omega_{\hat{\Phi}}, \quad t \geq t_0 \tag{5.34}$$

taking relations (5.2) and (5.3) into consideration, where

$$\hat{F}_e(\hat{e}, t) = H_2 F_e(e, t) = H_2 F_e(H_1 \hat{e} + H_0 \Delta\tilde{\hat{u}}_a, t), \quad \Delta\tilde{\hat{u}}_a = \tilde{\hat{u}}_a - \tilde{\hat{u}}_{ap} \tag{5.35}$$

is also asymptotically Lyapunov stable with an estimate $|\hat{e}(t)|$ of the transient $\hat{e}(t) = \hat{z}(t) - \hat{z}_p(t)$:

$$\begin{aligned}|\hat{e}(t)| &= |H_2 e(t)| \leq |H_2||e(t)| \leq |H_2|\mu|\Delta\Psi(e_0, t_0)|\exp[\gamma_0(t - t_0)] = \\ &= \hat{\mu}|\Delta\Psi(H_1\hat{e}_0 + H_0\Delta\tilde{\hat{u}}_{a0}, t_0)|\exp[\gamma_0(t - t_0)], \quad t \geq t_0\end{aligned} \tag{5.36}$$

where

$$\hat{\mu} = |H_2|\mu, \quad \hat{e}(t_0) + \hat{z}_p(t_0) = \hat{z}(t_0) = \hat{z}_0 \in \Omega_{\hat{\Psi}}$$

$$\Delta\tilde{\hat{u}}_{a0} = \Delta\tilde{\hat{u}}_a(t_0) = \tilde{\hat{u}}_a(t_0) - \tilde{\hat{u}}_{ap}(t_0)$$

and (because of inequality (5.10)) one obtains the prescribed precision

$$\hat{\varepsilon}_e = |H_2|\varepsilon_e > 0 \tag{5.37}$$

(where $\varepsilon_c > 0$ is the prescribed stabilization precision of the PM $z_p(t)$ (1.39), (1.38) of system (1.25)–(1.28), (1.24), (1.41)) of realization of the PM $\hat{z}_p(t)$ (5.33) and the prescribed attenuation time $T_p = t_p - t_0 > 0$ ($t_0 < t_p = t_0 + T_p < t_1$) of the transient $\hat{e} = \hat{z} - \hat{z}_p = H_2 e$, in such a way that the estimate

$$|\hat{e}(t)| \leq \hat{\varepsilon}_e, \quad \forall t \geq t_p = t_0 + T_p, \quad t_0 < t_p = t_0 + T_p < t_1 \tag{5.38}$$

is valid.

Next (proceeding as in the proof of Theorem 1), using the finite-increments formula for the vector function $\Delta\Phi_0(\hat{e}, t) = \Phi_0(\hat{e} + \hat{z}_p) - \Phi_0(\hat{z}_p)$ and relations (5.27), (5.28) and (5.36), we estimate the magnitude $|\bar{e}|$ of the vector $\bar{e} = \bar{z} - \bar{z}_p$ (5.32). We obtain

$$\begin{aligned}|\bar{e}| &= |\bar{z} - \bar{z}_p| = |\Phi_0(\hat{e} + \hat{z}_p) - \Phi_0(\hat{z}_p)| = |\Delta\Phi_0(\hat{e}, t)| = \\ &= \left|\left[\int_0^1 J_{\Delta\Phi_0}(s\hat{e}(t), t)ds\right]\hat{e}(t)\right| \leq \left|\int_0^1 J_{\Delta\Phi_0}(s\hat{e}(t), t)ds\right||\hat{e}(t)| \leq \hat{\mu}_0|\hat{e}(t)| \equiv \\ &\equiv \Delta\hat{\Phi}_0(\hat{e}(t)) \leq \hat{\mu}_0\hat{\mu}|\Delta\Psi(H_1\hat{e}_0, t_0) + H_0\Delta\tilde{\hat{u}}_{a0}, t_0|\exp[\gamma_0(t - t_0)] = \\ &= \beta|\Delta\Psi(H_1\Delta\Psi_0(\bar{e}_0, t_0) + H_0\Delta\tilde{\hat{u}}_{a0}, t_0)|\exp[\gamma_0(t - t_0)], \quad t \geq t_0\end{aligned} \tag{5.39}$$

where

$$\begin{aligned}&\bar{e}_0(t_0) + \bar{z}_p(t_0) = \bar{z}(t_0) = \bar{z}_0 \in \Omega_{\bar{\Psi}}, \ J_{\Delta\Phi_0}(\hat{e}, t) = \partial\Delta\Phi_0(\hat{e}, t)/(\partial\hat{e}) \\ &\sup_{\hat{\sigma}_e \in [0, \hat{e}], t \geq t_0} |J_{\Delta\Phi_0}(\hat{\sigma}_e, t)| = \hat{\mu}_0 < +\infty \\ &[0, e] = \{\hat{\eta}_e : \hat{\eta}_e = s\hat{e}, \hat{e} + \hat{z}_p = \hat{z} \in \Omega_{\hat{\Psi}}, 0 \leq s \leq 1\} \\ &\beta = \hat{\mu}_0\hat{\mu}, \quad \Delta\hat{\Phi}_0(0) \equiv 0, \quad \Delta\tilde{\hat{u}}_{a0} = \Delta\tilde{\hat{u}}_a(t_0) = \tilde{\hat{u}}_a(t_0) - \tilde{\hat{u}}_{ap}(t_0)\end{aligned} \tag{5.40}$$

and $0 < \hat{\mu}_0 < +\infty$ is some constant.

Relations (5.39) and (5.40) imply an estimate

$$|\bar{e}(t)| = |\Delta\Phi_0(\hat{e}(t), t)| \leq \Delta\hat{\Phi}_0(\hat{e}(t)) \equiv \hat{\mu}_0|\hat{e}(t)|, \quad t \geq t_0$$

from which it follows that the vector function $\Delta\Phi_0(\hat{e}, t)$ is continuous with respect to $\hat{e}$ at $\hat{e} = 0$, uniformly in $t \in [t, \infty)$, in such a way that $\Delta\Phi_0(0, t) \equiv 0$, so that the first assumption of the lemma is satisfied.

Hence, by the continuity of the vector function $\Delta\Psi_0(\bar{e}, t) = \Psi_0(\bar{e} + \bar{z}_p) - \Psi_0(\bar{z}_p)$, where $\Psi_0(\bar{z})$ has the form (1.16), (6.37), and since the solution $\hat{e} = 0$ of system (5.34), (5.35), (5.2), (5.3), (5.10) is asymptotically Lyapunov stable, it follows, again by the lemma, that the solution $\bar{e} = 0$ ($\bar{e} = \Delta\Phi_0(\hat{e}, t)$ and $\bar{e} = \Delta\Phi_0(0, t) = 0$) of the system

$$\dot{\bar{e}} = \bar{F}_e(\bar{e}, t), \quad \bar{e}(t_0) = \bar{e}_0, \quad \bar{e}_0(t_0) + \bar{z}_p(t_0) = \bar{z}(t_0) = \bar{z}_0 \in \Omega_{\bar{\Psi}}, \quad t \geq t_0 \tag{5.41}$$

because of the truth of (5.2) and (5.3), where

$$\begin{gathered}\bar{F}_e(\bar{e}, t) = (\Delta\Phi_0(\hat{e}, t))^{\cdot}\big|_{\hat{e} = \Delta\Psi_0(\bar{e}, t)} = J_{\Delta\Phi 0t}(\Delta\Psi_0(\bar{e}, t), t) + \\ + J_{\Delta\Phi 0\hat{e}}(\Delta\Psi_0(\bar{e}, t), t) \cdot \hat{F}_e(\Delta\Psi_0(\bar{e}, t), t) \\ J_{\Delta\Phi 0t}(\hat{e}, t) = \frac{\partial\Delta\Phi_0(\hat{e}, t)}{\partial t}, \quad J_{\Delta\Phi 0\hat{e}}(\hat{e}, t) = \frac{\partial\Delta\Phi_0(\hat{e}, t)}{\partial\hat{e}}\end{gathered} \tag{5.42}$$

is also asymptotically Lyapunov stable and (because of inequality (5.10)) one obtains the prescribed precision

$$\bar{\varepsilon}_e = \hat{\mu}_0\hat{\varepsilon}_e > 0 \tag{5.43}$$

(where $\hat{\varepsilon}_e$ (5.37) is the prescribed precision of stabilization of the PM $\hat{z}_p(t)$ (5.33) in the sp-model of TR dynamics (1.20)–(1.24), (1.15), (6.32), closed by the control law $\hat{u}_a$ (6.35), (5.31), (5.1)–(5.3), (5.10)) of realization of the PM $\bar{z}(t)$ (5.29), (5.30) and the prescribed attenuation time $T_p = t_p - t_0 > 0$ ($t_0 < t_p = t_0 + T_p < t_1$) of the transient $\bar{e} = \bar{z} - \bar{z}_p$, in such a way that the estimate

$$|\bar{e}(t)| \leq \bar{\varepsilon}_e, \quad \forall t \geq t_p = t_0 + T_p, \quad t_0 < t_p = t_0 + T_p < t_1 \tag{5.44}$$

is valid.

We have thus proved the following theorem.

Theorem 2. Let $\bar{z}_p(t)$ (5.29), (5.30) be a given (constructed) PM for the original model of TR motion (1.9)–(1.13), (1.8).

Then the stabilizing control law u_a (5.31), (5.1)–(5.3), (5.10) guarantees asymptotic stability of the PM $\bar{z}_p(t)$ (5.29), (5.30); the magnitude $|\bar{e}(t)|$ of the transient $\bar{e}(t) = \bar{z}(t) - \bar{z}_p(t)$ in the closed original model of TR motion (1.9)–(1.13), (5.31), (5.1)–(5.3), (5.10) satisfies the estimate (5.39), (5.40); the prescribed precision $\bar{\varepsilon}_e$ (5.43) of realization of the PM $\bar{z}_p(t)$ (5.29), (5.30) and attenuation time $T_p = t_p - t_0 > 0$ ($t_0 < t_p = t_0 + T_p < t_1$) of the transient $\bar{e}(t) = \bar{z}(t) - \bar{z}_p(t)$ are guaranteed, in such a way that the estimate (5.44) is valid.

Note that, in order to find the above-mentioned estimates for $|e(t)|$ (5.27), (5.28) and $|\bar{e}(t)|$ (5.39), (5.40), of the solutions $e(t)$ and $\bar{e}(t)$ of systems (5.25), (5.26) and (5.41), (5.42), respectively, one can also use the technique described in [11, pp. 921–928].

6. APPENDIX

We first apply to the equations of the original model of TR motion (1.9)–(1.13), (1.8) non-linear one-to-one continuously differentiable transformations of the state space coordinates $\bar{z}$ (1.10) and

$$\begin{gathered}\tilde{z} = \mathrm{col}(\tilde{z}_1, \tilde{z}_2, \tilde{z}_3, \tilde{z}_4) \\ (\tilde{z}_1 = \mathrm{col}(x_B, y_B), \quad \tilde{z}_2 = \bar{z}_2 = \mathrm{col}(\psi_c, \theta), \quad \tilde{z}_3 = \dot{\tilde{z}}_2 = \bar{z}_3 = \mathrm{col}(\dot{\psi}_c, \dot{\theta}), \quad \tilde{z}_4 = \tilde{I}_a)\end{gathered} \tag{6.1}$$

and controls u_a (1.8) and

$$\bar{u}_a = \mathrm{col}(\bar{u}_{a1}, \bar{u}_{a2}) \tag{6.2}$$

by the formulae

$$\tilde{z} = \bar{\Psi}(\bar{z}), \quad \bar{z} \in \Omega_{\bar{\Psi}} \tag{6.3}$$

$$\bar{z} = \bar{\Psi}^{-1}(\tilde{z}) \equiv \bar{\Phi}(\tilde{z}), \quad \tilde{z} \in \Omega_{\bar{\Phi}} \tag{6.4}$$

and

$$\bar{u}_a = \bar{\Psi}_5(\bar{z}_{22}^4, u_a) \tag{6.5}$$

$$u_a = \bar{\Psi}_5^{-1}(\bar{z}_{22}^4, \bar{u}_a) = \bar{\Psi}_5^{-1}(\bar{\Phi}_{22}^4(\tilde{z}_{22}^4), \bar{u}_a) \equiv \bar{\Phi}_5(\tilde{z}_{22}^4, \bar{u}_a) \tag{6.6}$$

respectively, where

$$\bar{z}_{22}^4 = \mathrm{col}(\bar{z}_{22}, \bar{z}_3, \bar{z}_4), \quad \tilde{z}_{22}^4 = \mathrm{col}(\tilde{z}_{22}, \tilde{z}_3, \tilde{z}_4)$$

Here

$$\bar{\Phi}_{22}^4(\tilde{z}_{22}^4) = \mathrm{col}(\bar{\Phi}_{22}(\tilde{z}_{22}), \bar{\Phi}_3(\tilde{z}_3), \bar{\Phi}_4(\tilde{z}_{22}^4)), \quad \bar{\Phi}_2(\tilde{z}_2) = \mathrm{col}(\bar{\Phi}_{21}(\tilde{z}_{21}), \bar{\Phi}_{22}(\tilde{z}_{22})) \tag{6.7}$$

$$\bar{\Psi}(\bar{z}) = \mathrm{col}(\bar{\Psi}_1(\bar{z}_1, \bar{z}_{21}), \bar{\Psi}_2(\bar{z}_2), \bar{\Psi}_3(\bar{z}_3), \bar{\Psi}_4(\bar{z}_{22}^4))$$

$$(\bar{\Psi}_1(\bar{z}_1, \bar{z}_{21}) = \mathrm{col}(\bar{z}_{11} - l_2 \cos\bar{z}_{21}, \bar{z}_{12} - l_2 \sin\bar{z}_{21})$$

$$\bar{\Psi}_2(\bar{z}_2) = \bar{z}_2, \quad \bar{\Psi}_3(\bar{z}_3) = \dot{\bar{\Psi}}_2(z_2) = \dot{\bar{z}}_2 = \bar{z}_3$$

$$\bar{\Psi}_4(\bar{z}_{22}^4) = \dot{\bar{\Psi}}_3(\bar{z}_3) = \dot{\bar{z}}_3 = \bar{F}_3(\bar{z}_{22}^4) =$$

$$= \bar{C}_3(\bar{z}_{22}^3) + \bar{D}_3(\bar{z}_{22})\bar{z}_4 = \bar{K}_4(\bar{z}_{22}^3) + \bar{L}_4(\bar{z}_{22})\bar{z}_4$$

$$\bar{K}_4(\bar{z}_{22}^3) = \bar{C}_3(\bar{z}_{22}^3), \quad \bar{L}_4(\bar{z}_{22}) = \bar{D}_3(\bar{z}_{22}), \quad \bar{z}_{22}^3 = \mathrm{col}(\bar{z}_{22}, \bar{z}_3)) \tag{6.8}$$

$$\bar{\Phi}(\tilde{z}) = \mathrm{col}(\bar{\Phi}_1(\tilde{z}_1, \tilde{z}_{21}), \bar{\Phi}_2(\tilde{z}_2), \bar{\Phi}_3(\tilde{z}_3), \bar{\Phi}_4(\tilde{z}_{22}^4))$$

$$(\bar{\Phi}_1(\tilde{z}_1, \tilde{z}_{21}) = \mathrm{col}(\tilde{z}_{11} + l_2 \cos\tilde{z}_{21}, \tilde{z}_{12} + l_2 \sin\tilde{z}_{21})$$

$$\bar{\Phi}_2(\tilde{z}_2) = \tilde{z}_2, \quad \bar{\Phi}_3(\tilde{z}_3) = \tilde{z}_3, \quad \bar{\Phi}_4(\tilde{z}_{22}^4) = \bar{M}_4(\tilde{z}_{22}^3) + \bar{N}_4(\tilde{z}_{22})\tilde{z}_4$$

$$\bar{M}_4(\tilde{z}_{22}^3) = -\bar{N}_4(\tilde{z}_{22})\bar{K}_4(\tilde{z}_{22}^3), \quad \bar{N}_4(\tilde{z}_{22}) = \bar{L}_4^{-1}(\tilde{z}_{22})$$

$$\tilde{z}_{22}^3 = \mathrm{col}(\tilde{z}_{22}, \tilde{z}_3) = \bar{z}_{22}^3, \quad \tilde{z}_{22} = \bar{z}_{22}, \quad \tilde{z}_3 = \bar{z}_3)$$

are vector functions defined on the respective sets

$$\Omega_{\bar{\Psi}} = \begin{cases} \Omega_{\bar{\Psi}}^+, & \text{if} \quad \theta \in \Omega_\theta^+ = \{\theta \in R^1 : 0 < \varepsilon_\theta < \theta < \pi/2 - \varepsilon_\theta\} \\ \Omega_{\bar{\Psi}}^-, & \text{if} \quad \theta \in \Omega_\theta^- = \{\theta \in R^1 : -\pi/2 + \varepsilon_\theta < \theta < -\varepsilon_\theta < 0\} \end{cases} \tag{6.9}$$

where $\varepsilon_\theta > 0$ is some real number; the sets in this definition are

$$\begin{aligned} \Omega_{\bar{\Psi}}^\pm &= \{\bar{z} = \mathrm{col}(x_c, y_c, \psi_c, \theta, \dot{\psi}_c, \dot{\theta}, I_a) \in R^8 : \theta \in \Omega_\theta^\pm\} \\ \Omega_{\bar{\Phi}} &= \{\tilde{z} = \mathrm{col}(x_B, y_B, \psi_c, \theta, \dot{\psi}_c, \dot{\theta}, \tilde{I}_a) = \bar{\Psi}(\bar{z}) \in R^8 : \bar{z} \in \Omega_{\bar{\Psi}}\} \end{aligned} \tag{6.10}$$

while the vector functions

$$\bar{\Psi}_5(\bar{z}^4_{22}, u_a) = \dot{\bar{\Psi}}_4(\bar{z}^4_{22}) = \frac{\partial\bar{\Psi}_4(\bar{z}^4_{22})}{\partial\bar{z}_{22}}\bar{z}_{32} + \frac{\partial\bar{\Psi}_4(\bar{z}^4_{22})}{\partial\bar{z}_3}\bar{F}_3(\bar{z}^4_{22}) + \frac{\partial\bar{\Psi}_4(\bar{z}^4_{22})}{\partial\bar{z}_4}\bar{F}_4(\bar{z}^4_{22}, u_a) = \bar{K}_5(\bar{z}^4_{22}) + \bar{L}_5(\bar{z}_{22})u_a \tag{6.11}$$

$$(\bar{K}_5(\bar{z}^4_{22}) = \frac{\partial\bar{\Psi}_4(\bar{z}^4_{22})}{\partial\bar{z}_{22}}\bar{z}_{32} + \frac{\partial\bar{\Psi}_4(\bar{z}^4_{22})}{\partial\bar{z}_3}\bar{F}_3(\bar{z}^4_{22}) + \frac{\partial\bar{\Psi}_4(\bar{z}^4_{22})}{\partial\bar{z}_4}\bar{C}_4(\bar{z}^4_{22})$$

$$\bar{L}_5(\bar{z}_{22}) = \frac{\partial\bar{\Psi}_4(\bar{z}^4_{22})}{\partial\bar{z}_4}\bar{D}_4 = \bar{L}_4(\bar{z}_{22})\bar{D}_4) \tag{6.12}$$

$$\bar{\Phi}_5(\tilde{z}^4_{22}, \bar{u}_a) = \bar{M}_5(\tilde{z}^4_{22}) + \bar{N}_5(\tilde{z}_{22})\bar{u}_a$$

$$(\bar{M}_5(\tilde{z}^4_{22}) = -\bar{N}_5(\tilde{z}_{22})\bar{K}_5(\bar{\Phi}^4_{22}(\tilde{z}^4_{22})), \bar{N}_5(\tilde{z}_{22}) = \bar{L}_5^{-1}(\tilde{z}_{22}))$$

are defined respectively on the sets

$$\Omega_{\bar{\Psi}5} = \{(\bar{z}^4_{22}, u_a) : \bar{z}^4_{22} = \mathrm{col}(\bar{z}_{22}, \bar{z}_3, \bar{z}_4) = \mathrm{col}(\theta, \psi_c, \dot{\theta}, I_a) \in \Omega_{\bar{\Psi}}, u_a \in R^2\} \tag{6.13}$$

$$\Omega_{\bar{\Phi}5} = \{(\tilde{z}^4_{22}, \bar{u}_a) : \tilde{z}^4_{22} = \mathrm{col}(\tilde{z}_{22}, \tilde{z}_3, \tilde{z}_4) = \mathrm{col}(\theta, \psi_c, \dot{\theta}, \tilde{I}_a) \in \Omega_{\bar{\Phi}}, \quad \bar{u}_a = \bar{\Psi}_5(\bar{z}^4_{22}, u_a) \in R^2, \quad (\bar{z}^4_{22}, u_a) \in \Omega_{\bar{\Psi}5}\} \tag{6.14}$$

Then the equations of the original model of TR motion (1.9)–(1.13), (1.8) (after application of the transformations (6.3)–(6.14)) may be written as a system of non-linear ODEs

$$\dot{\tilde{z}} = \tilde{F}(\tilde{z}, \bar{u}_a), \quad \tilde{z}_0 = \tilde{z}(t_0), \quad t \geq t_0 \tag{6.15}$$

where $\tilde{z}$ is the state vector (1.14) of the system, and

$$\tilde{F}(\tilde{z}, \bar{u}_a) = J_{\bar{\Psi}}(\bar{\Phi}(\tilde{z})) \cdot \bar{F}(\bar{\Phi}(\tilde{z}), \bar{\Phi}_5(\tilde{z}^4_{22}, \bar{u}_a)) = \mathrm{col}(\tilde{F}_1(\tilde{z}_2, \tilde{z}_{31}), \tilde{F}_2(\tilde{z}_3), \tilde{F}_3(\tilde{z}_4), \tilde{F}_4(\bar{u}_a)), \quad J_{\bar{\Psi}}(\bar{z}) = \partial\bar{\Psi}(\bar{z})/(\partial\bar{z}) \tag{6.16}$$

$$\tilde{F}_1(\tilde{z}_2, \tilde{z}_{31}) = \mathrm{col}(\tilde{z}_{31} l \,\mathrm{ctg}\,\tilde{z}_{22} \cos\tilde{z}_{21}, \tilde{z}_{31} l \,\mathrm{ctg}\,\tilde{z}_{22} \sin\tilde{z}_{21})$$
$$\tilde{F}_2(\tilde{z}_3) = \tilde{z}_3, \quad \tilde{F}_3(\tilde{z}_4) = \tilde{z}_4, \quad \tilde{F}_4(\bar{u}_a) = \bar{u}_a \tag{6.17}$$

are vector functions.

We then apply to system (6.15), (6.16), (6.17), (6.1), (6.2) non-linear one-to-one continuously differentiable transformations of the state space coordinates $\tilde{z}$ (6.1) and

$$\hat{z} = \mathrm{col}(\hat{z}_1, \hat{z}_2, \hat{z}_3, \hat{z}_4) \quad (\hat{z}_1 = \mathrm{col}(x_B, y_B)$$
$$\hat{z}_2 = \mathrm{col}(\psi_c, \kappa_B), \quad \hat{z}_3 = \mathrm{col}(V_B, \dot{\kappa}_B), \quad \hat{z}_4 = \hat{I}_a) \tag{6.18}$$

and of the controls $\bar{u}_a$ (6.2) and

$$\hat{u}_a = \mathrm{col}(\hat{u}_{a1}, \hat{u}_{a2}) \tag{6.19}$$

by formulae

$$\hat{z} = \hat{\Psi}(\tilde{z}), \quad \tilde{z} \in \Omega_{\hat{\Psi}} \tag{6.20}$$

$$\tilde{z} = \hat{\Psi}^{-1}(\hat{z}) \equiv \hat{\Phi}(\hat{z}), \quad \hat{z} \in \Omega_{\hat{\Phi}} \tag{6.21}$$

and

$$\hat{u}_a = \hat{\Psi}_5(\tilde{z}_{22}^4, \tilde{u}_a) \tag{6.22}$$

$$\tilde{u}_a = \hat{\Psi}_5^{-1}(\tilde{z}_2^4, \hat{u}_a) = \hat{\Psi}_5^{-1}(\hat{\Phi}_{22}^4(\hat{z}_{22}^4), \hat{u}_a) = \hat{\Phi}_5(\hat{z}_{22}^4, \hat{u}_a) \tag{6.23}$$

respectively, where $\hat{z}_{22}^4 = \mathrm{col}(\hat{z}_{22}, \hat{z}_3, \hat{z}_4)$,

$$\begin{aligned}
&\hat{\Psi}(\tilde{z}) = \mathrm{col}(\hat{\Psi}_1(\tilde{z}_1), \hat{\Psi}_2(\tilde{z}_2), \hat{\Psi}_3(\tilde{z}_{22}, \tilde{z}_3), \hat{\Psi}_4(\tilde{z}_{22}^4))\\
&(\hat{\Psi}_1(\tilde{z}_1) = \tilde{z}_1, \quad \hat{\Psi}_2(\tilde{z}_2) = \mathrm{col}(\tilde{z}_{21}, l^{-1}\,\mathrm{ctg}\,\tilde{z}_{22})\\
&\hat{\Psi}_3(\tilde{z}_{22}^4) = \mathrm{col}(\hat{\Psi}_{31}(\tilde{z}_{22}^{31}), \hat{\Psi}_{32}(\tilde{z}_{22}, \tilde{z}_{32}))\\
&\hat{\Psi}_{31}(\tilde{z}_{22}^{31}) = \tilde{z}_{31} l\,\mathrm{ctg}\,\tilde{z}_{22}\\
&\hat{\Psi}_{32}(\tilde{z}_{22}, \tilde{z}_{32}) = l^{-1}(1 + \mathrm{ctg}^2\tilde{z}_{22})\tilde{z}_{32}\\
&\hat{\Psi}_4(\tilde{z}_{22}^4) = \dot{\hat{\Psi}}_3(\tilde{z}_{22}, \tilde{z}_3) = \hat{K}_4(\tilde{z}_{22}^3) + \hat{L}_4(\tilde{z}_{22})\tilde{z}_4\\
&\hat{K}_4(\tilde{z}_{22}^3) = \frac{\partial \hat{\Psi}_3(\tilde{z}_{22}^3)}{\partial \tilde{z}_{22}}\tilde{z}_{32} = \begin{Vmatrix} -\dfrac{l}{\sin^2\tilde{z}_{22}}\tilde{z}_{31} \\ \dfrac{2\,\mathrm{tg}\,\tilde{z}_{22}}{l\cos^2\tilde{z}_{22}}\tilde{z}_{32} \end{Vmatrix}\tilde{z}_{32}\\
&\hat{L}_4(\tilde{z}_{22}) = \frac{\partial \tilde{\Psi}_3(\tilde{z}_{22}^3)}{\partial \tilde{z}_3} = \mathrm{diag}(l\,\mathrm{ctg}\,\tilde{z}_{22}, l^{-1}(1 + \mathrm{tg}^2\tilde{z}_{22}))
\end{aligned} \tag{6.24}$$

$$\begin{aligned}
&\hat{\Phi}(\hat{z}) = \mathrm{col}(\hat{\Phi}_1(\hat{z}_1), \hat{\Phi}_2(\hat{z}_2), \hat{\Phi}_3(\hat{z}_{22}, \hat{z}_3), \hat{\Phi}_4(\hat{z}_{22}^4))\\
&(\hat{\Phi}_1(\hat{z}_1) = \hat{z}_1 = \mathrm{col}(x_B, y_B), \hat{\Phi}_2(\hat{z}_2) = \mathrm{col}(\hat{\Phi}_{21}(\hat{z}_{21}), \hat{\Phi}_{22}(\hat{z}_{22}))\\
&\hat{\Phi}_{21}(\hat{z}_{21}) = \hat{z}_{21}, \quad \hat{\Phi}_{22}(\hat{z}_{22}) = \mathrm{arctg}(l\hat{z}_{22})\\
&\hat{\Phi}_3(\hat{z}_{22}^3) = \mathrm{col}(\hat{\Phi}_{31}(\hat{z}_{22}^{31}), \hat{\Phi}_{32}(\hat{z}_{22}, \hat{z}_{32}))\\
&\hat{\Phi}_{31}(\hat{z}_{22}^{31}) = \hat{z}_{22}\hat{z}_{31}, \quad \hat{\Phi}_{32}(\hat{z}_{22}, \hat{z}_{32}) = \hat{z}_{32} l/(1 + \hat{z}_{22}^2 l^2)\\
&\hat{\Phi}_4(\hat{z}_{22}^4) = \hat{M}_4(\hat{z}_{22}^3) + \hat{N}_4(\hat{z}_{22})\hat{z}_4, \quad \hat{M}_4(\hat{z}_{22}^3) = -\hat{N}_4(\hat{z}_{22})\hat{K}_4(\hat{\Phi}_{22}^3(\hat{z}_{22}^3))\\
&\hat{N}_4(\hat{z}_{22}) = \hat{L}_4^{-1}(\hat{\Phi}_{22}(\hat{z}_{22})), \quad \hat{\Phi}_{22}^3(\hat{z}_{22}^3) = \mathrm{col}(\hat{\Phi}_{22}(\hat{z}_{22}), \hat{\Phi}_3(\hat{z}_{22}^3))\\
&\hat{z}_{22}^3 = \mathrm{col}(\hat{z}_{22}, \hat{z}_3), \quad \hat{z}_{22}^{31} = \mathrm{col}(\hat{z}_{22}, \hat{z}_{31}))
\end{aligned} \tag{6.25}$$

are vector functions defined on the respective sets

$$\Omega_{\Psi} = \Omega_{\overline{\Phi}} \tag{6.26}$$

$$\Omega_{\Phi} = \{\hat{z} = \hat{\Psi}(\tilde{z}) \in R^8 : \tilde{z} \in \Omega_{\Psi}\} \tag{6.27}$$

while the vector functions

$$\hat{\Psi}_5(\tilde{z}_{22}^4, \tilde{u}_a) = \dot{\hat{\Psi}}_4(\tilde{z}_{22}^4) = \hat{K}_5(\tilde{z}_{22}^4) + \hat{L}_5(\tilde{z}_{22})\tilde{u}_a \tag{6.28}$$

$$\left(\hat{K}_5(\tilde{z}_{22}^4) = \frac{\partial \hat{\Psi}_4(\tilde{z}_{22}^4)}{\partial \tilde{z}_{22}}\tilde{z}_{32} + \frac{\partial \hat{\Psi}_4(\tilde{z}_{22}^4)}{\partial \tilde{z}_3}\tilde{z}_4, \quad \hat{L}_5(\tilde{z}_{22}) = \frac{\partial \hat{\Psi}_4(\tilde{z}_{22}^4)}{\partial \tilde{z}_4} = \hat{L}_4(\tilde{z}_{22})\right)$$

$$\begin{aligned}&\hat{\Phi}_5(\hat{z}_{22}^4, \hat{u}_a) = \hat{M}_5(\hat{z}_{22}^4) + \hat{N}_5(\hat{z}_{22})\hat{u}_a \\ &(\hat{M}_5(\hat{z}_{22}^4) = -\hat{N}_5(\hat{z}_{22})\hat{K}_5(\hat{\Phi}_{22}^4(\hat{z}_{22}^4)), \quad \hat{N}_5(\hat{z}_{22}) = \hat{L}_5^{-1}(\hat{\Phi}_{22}(\hat{z}_{22})) \\ &\hat{\Phi}_{22}^4(\hat{z}_{22}^4) = \mathrm{col}(\hat{\Phi}_{22}(\hat{z}_{22}), \hat{\Phi}_3(\hat{z}_{22}^3), \hat{\Phi}_4(\hat{z}_{22}^4)))\end{aligned} \tag{6.29}$$

are defined on the respective sets

$$\Omega_{\hat{\Psi}5} = \Omega_{\bar{\Phi}5} \tag{6.30}$$

$$\Omega_{\hat{\Phi}5} = \left\{(\hat{z}_{22}^4, \hat{u}_a) : \hat{z}_{22}^4 = \hat{\Psi}_{22}^4(\bar{z}_{22}^4), \hat{u}_a = \hat{\Psi}_5(\bar{z}_{22}^4, \bar{u}_a), (\bar{z}_{22}^4, \bar{u}_a) \in \Omega_{\bar{\Phi}5}\right\} \tag{6.31}$$

Then the equations of TR motion (6.15), (6.16), (6.1), (6.2) (after application of the transformations (6.20)–(6.31)) are reduced to a system of non-linear ODEs of the special form (1.20), (1.14), (1.15), where

$$\hat{F}(\hat{z}, \hat{u}_a) = J_{\hat{\Psi}}(\hat{\Phi}(\hat{z})) \cdot \tilde{F}(\hat{\Phi}(\hat{z}), \hat{\Phi}_5(\hat{z}_{22}^4, \hat{u}_a)) \tag{6.32}$$

is a vector function and the function $J_{\hat{\Psi}}(\bar{z}) = \partial\hat{\Psi}(\bar{z})/(\partial\bar{z})$ has the representation (1.21), (1.22).

Thus, the non-linear one-to-one continuously differentiable transformations of the state space coordinates $\bar{z}$ (1.10) and $\hat{z}$ (6.18) and of the controls u_a and $\hat{u}_a$ (6.19) by formulae (1.16) and (1.17), which have the form

$$\hat{z} = \Psi_0(\bar{z}), \quad \bar{z} \in \Omega_{\Psi 0} \tag{6.33}$$

$$\bar{z} = \Psi_0^{-1}(\hat{z}) \equiv \Phi_0(\hat{z}), \quad \hat{z} \in \Omega_{\Phi 0} \tag{6.34}$$

and formulae (1.18) and (1.19), which have the form

$$\hat{u}_a = \Psi_{05}(\bar{z}_2^4, u_a) \tag{6.35}$$

$$u_a = \Phi_{05}(\bar{z}_2^4, \hat{u}_a) \tag{6.36}$$

respectively, where

$$\begin{aligned}&\bar{z}_2^4 = \mathrm{col}(\bar{z}_2, \bar{z}_3, \bar{z}_4) = \mathrm{col}(\psi_c, \theta, \dot{\psi}_c, \dot{\theta}, I_a), \quad \hat{z}_2^4 = \mathrm{col}(\hat{z}_2, \hat{z}_3, \hat{z}_4) = \\ &= \mathrm{col}(\psi_c, \kappa_B, V_B, \dot{\kappa}_B, \hat{I}_a), \quad \Phi_{02}^4(\hat{z}_2^4) = \mathrm{col}(\Phi_{02}(\hat{z}_2), \Phi_{03}(\hat{z}_2^3), \Phi_{04}(\hat{z}_2^4))\end{aligned} \tag{6.37}$$

$$\begin{aligned}&\Psi_0(\bar{z}) = \hat{\Psi}(\bar{\Psi}(\bar{z})) \\ &\Phi_0(\hat{z}) \equiv \Psi_0^{-1}(\hat{z}) = \bar{\Phi}(\hat{\Phi}(\hat{z}))\end{aligned} \tag{6.38}$$

are vector functions defined on the respective sets

$$\Omega_{\Psi 0} = \Omega_{\bar{\Psi}} \tag{6.39}$$

$$\Omega_{\Phi 0} = \Omega_{\hat{\Phi}} \tag{6.40}$$

and

$$\Psi_{05}(\bar{z}_{22}^4, u_a) = \hat{u}_a = \hat{\Psi}_5(\bar{z}_{22}^4, \bar{u}_a) = \hat{\Psi}_5(\bar{\Psi}_{22}^4(\bar{z}_{22}^4), \bar{\Psi}_5(\bar{z}_{22}^4, u_a)) \tag{6.41}$$

are vector functions where

$$\begin{aligned}&\bar{\Psi}_{22}^4(\bar{z}_{22}^4) = \mathrm{col}(\bar{\Psi}_{22}(\bar{z}_{22}), \bar{\Psi}_3(\bar{z}_{22}^3), \bar{\Psi}_4(\bar{z}_{22}^4)) \\ &\Phi_{05}(\hat{z}_{22}^4, \hat{u}_a) = u_a \equiv \Psi_{05}^{-1}(\hat{z}_{22}^4, \hat{u}_a) = \Psi_{05}^{-1}(\Phi_{022}^4(\hat{z}_{22}^4), \hat{u}_a)\end{aligned} \tag{6.42}$$

are defined on the respective sets

$$\Omega_{\Psi_{05}} = \{(\bar{z}_{22}^4, u_a) : \bar{z}_{22}^4 \in \Omega_{\bar{\Psi}}, u_a \in R^2\} \tag{6.43}$$

$$\Omega_{\Phi_{05}} = \{(\hat{z}_{22}^4, \hat{u}_a) : \hat{u}_a = \Psi_{05}(\bar{z}_{22}^4, u_a) = \Psi_{05}(\Phi_{022}^4(\hat{z}_{22}^4), u_a), \hat{z}_{22}^4 \in \Omega_{\hat{\Phi}}, u_a \in R^2\} \tag{6.44}$$

reduce the original equations of the model of TR motion (1.9)–(1.13), (1.8) to a system of non-linear ODEs of the special form (1.20)–(1.22), (1.14), (1.15).

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